1 Objective

We show how the first and the second derivative of a function can be used to describe the qualitative properties of its graph. This will be used in the next lecture to solve optimization problems.

2 Roots, peaks and troughs, inflection points

2.1 Roots: \( f(x) = 0 \)

A solution to the equation \( f(x) = 0 \) where \( f \) is a function is called a root or \( x \)-intercept. At a root the function may or may not switch sign. In the figure on the left the function has a root at \( x = -1 \) and switches sign there. In the figure on the right the function has a root at \( x = -1 \), but does not switch sign there.

\[
\begin{align*}
\text{f}(x) &= x^3 + 1 \\
\text{f}'(x) &= 3x^2
\end{align*}
\]

\[
\begin{align*}
\text{f}(x) &= x^2 \\
\text{f}'(x) &= 2x
\end{align*}
\]

**Lemma 1** Suppose the function \( y = f(x) \) is differentiable on the interval \( a \leq x \leq b \) with \( a < b \). Suppose \( x_0 \) with \( a < x_0 < b \) is a root of the a function and \( f'(x_0) \neq 0 \). Then the function switches sign at \( x_0 \).

2.2 Critical points: \( f'(x) = 0 \)

A solution to the equation \( f'(x) = 0 \), where \( f' \) is the first derivative of a function \( f \), is called a critical point or a stationary point of the function \( f \). At a stationary point the
graph of the function has a horizontal tangent.

\[ x = 0 \] is a critical point

Stationary points come in three varieties:

Suppose the differentiable function is defined on an interval \( a \leq x \leq b \) with \( a < b \) and that \( a < x_0 < b \) is a critical point of the function.

- We call \( x_0 \) a peak or a relative maximum or a local maximum if the function turns from being increasing on the left to being decreasing on the right at \( x_0 \).
- We call \( x_0 \) a trough or a relative minimum or a local minimum if the function turns from being decreasing on the left to being increasing on the right at \( x_0 \).
- We call \( x_0 \) a saddle point if the function is increasing (or decreasing) both to the left and to the right of \( x_0 \).

In the following graphs \( x_0 = 0 \) is a critical point. The first row shows the graphs of the functions. The second row shows the graphs of the derivative.

A peak, \( f(x) = -x^4 + 1 \)  
A trough, \( f(x) = x^4 - 1 \)  
A saddlepoint, \( f(x) = -x^3 + 1 \)

\[
\begin{align*}
  f'(x) &= -4x^3 \\
  f'(x) &= 4x^3 \\
  f'(x) &= -3x^2
\end{align*}
\]

**Proposition 1** Suppose the function \( y = f(x) \) is continuously differentiable on the interval \( a \leq x \leq b \) with \( a < b \) and that \( a < x_0 < b \) is a critical point of the function. Then
a) $x_0$ is a peak of the function if and only if the first derivative $f'(x)$ changes sign from $+$ to $-$ at $x_0$.

b) $x_0$ is a trough of the function if and only if the first derivative $f'(x)$ changes sign from $-$ to $+$ at $x_0$.

2.2.1 A sufficient criterion for peaks and troughs

Often the second derivative can be used to determine whether a critical point $x_0$ is a peak or a trough of the function.

**Question:** Suppose $f'(x) = 0$. Is $x_0$ a peak, a trough or a saddle point if a) $f''(x_0) > 0$, b) $f''(x_0) < 0$ or c) $f''(x_0) = 0$?

**Correct answer:**

to a) The function has a trough at $x_0$ (because it is upward bowed near $x_0$).
to b) The function has a peak at $x_0$ (because it is downward bowed near $x_0$).
to b) I don’t know. It could be a trough a peak or a saddle point. Ask Dieter.

**Exercise 1** a) Verify that $x = 2$ is a critical point of the function

\[ y = f(x) = 3x^4 - 16x^3 + 30x^2 - 24x + 5. \]

Is $x = 2$ a trough, a peak or a saddle point?

b) Verify that $x = 1$ is a critical point of the function

\[ y = f(x) = 3x^4 - 16x^3 + 30x^2 - 24x + 5. \]

Is $x = 1$ a trough, a peak or a saddle point?

2.3 Candidates for inflection points: $f''(x) = 0$

$x_0$ is an *inflection point* of the function $f(x)$ if the function changes from being concave to being convex or vice versa at $x_0$. For instance, $f(x) = x(x+1)(x-1)$ has an inflection point at $x_0 = 0$.

\[ f(x) = x(x+1)(x-1) \]

We call $x_0$ a *candidate for an inflection point* if $f''(x_0) = 0$.

**Proposition 2** Suppose the function $y = f(x)$ is twice continuously differentiable on the interval $a \leq x \leq b$ with $a < b$ and that $a < x_0 < b$ is a candidate for an inflection point, i.e., $f''(x_0) = 0$. Then $x_0$ is indeed an inflection point if and only if the second derivative changes sign at $x_0$.

**Proposition 3** Suppose the function $y = f(x)$ is three times differentiable on the interval $a \leq x \leq b$ with $a < b$. Suppose that $a < x_0 < b$ satisfies $f''(x_0) = 0$ and $f'''(x_0) \neq 0$. Then $x_0$ is an inflection point.
3 An odd polynomial

We will study the following function

\[ y = f(x) = -\frac{1}{5}x^5 + \frac{13}{3}x^3 - 36x \]

Such a polynomial is called *odd* because it contains no powers with even index like \( x^4, \) \( x^2 \) and \( x^0 = 1. \)

4 The roots

Clearly, \( x = 0 \) is a root and hence \( x \) a factor, so

\[ y = f(x) = -\frac{1}{5}x \left( x^4 - \frac{65}{3}x^2 + 180 \right) \]

We claim that the fourth order polynomial

\[ g(x) = x^4 - \frac{65}{3}x^2 + 180 \]

has no roots and is everywhere strictly positive. Hence \( f(x) \) has only a root at \( x = 0, \) is positive to the left and negative to the right.

To see this, compare \( g(x) \) with the quadratic polynomial

\[ z^2 - \frac{65}{3}z + 180 \]

The latter has no roots because the discriminant is negative \( (-\frac{65}{3})^2 - 4 \times 180 \approx -250.56 < 0. \) If \( x \) were a root of \( g(x) \) then \( z = x^2 \) would be a root of the quadratic polynomial, which is impossible. So \( g(x) \) has no root. Moreover, \( g(0) = 180 > 0. \) If there would exist a number \( x_1 \) with \( g(x_1) < 0 \) then (since \( g(x) \) is differentiable and hence continuous) there would be a number between \( x = 0 \) and \( x_1 \) where the graph of \( g(x) \) cuts the horizontal axis. Again this is impossible because \( g(x) \) has no root. Therefore \( g(x) > 0 \) for all numbers \( x. \)

4.1 The critical points

The first derivative of \( f(x) \) is

\[ y' = f'(x) = \left( -\frac{1}{5}x^5 + \frac{13}{3}x^3 - 36x \right)' = -x^4 + 13x^2 - 36 \]

Compare this polynomial with the quadratic polynomial

\[ -z^2 + 13z - 36 \]
If we replace in the latter \( z \) with \( x^2 \) we obtain our above derivative. It is easily seen that \( z_0 = 4 \) and \( z_1 = 9 \) are the roots of the quadratic polynomial. Therefore we have the factorization

\[-z^2 + 13z - 36 = -(z - 4)(z - 9).\]

Consequently,

\[f'(x) = -x^4 + 13x^2 - 36 = -(x^2 - 4)(x^2 - 9)\]

(You should verify this by expanding the product on the right.) \( x^2 - 4 \) is a difference of two squares and therefore \( x^2 - 4 = (x + 2)(x - 2) \). Using the same trick on \( x^2 - 9 \) we obtain the factorization

\[y' = f'(x) = -(x + 2)(x - 2)(x + 3)(x - 3)\]

We conclude that our polynomial \( f(x) \) has four critical points, namely \( x = -3, x = -2, x = +2 \) and \( x = +3 \). To see where our function is increasing / decreasing etc. we draw a sign diagram for the derivative.

Our sign diagram contains one row for each factor and one row for \( f'(x) \) itself. It contains one column for each root and additional columns for the intervals between the roots. It also contains a column for all numbers to the left of all roots and a column for all numbers to the right.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x &lt; -3 )</th>
<th>( x = -3 )</th>
<th>( -3 &lt; x &lt; -2 )</th>
<th>( x = -2 )</th>
<th>( -2 &lt; x &lt; 2 )</th>
<th>( x = 2 )</th>
<th>( 2 &lt; x &lt; 3 )</th>
<th>( x = 3 )</th>
<th>( 3 &lt; x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -1 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
<td>( x + 3 )</td>
<td>-</td>
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<td>-</td>
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<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( x + 2 )</td>
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<td>-</td>
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<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( x - 2 )</td>
<td>-</td>
<td>-</td>
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<td>-</td>
<td>-</td>
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<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( x - 3 )</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The last line of the diagram tells us precisely where the derivative is negative positive or zero. It hence tells us where the function \( f(x) \) is increasing. For instance, in the interval or graph of the function is above, below and on the horizontal axis. For instance, in the interval \(-2 < x < 2\) the first derivative is negative and the function hence decreasing.

The diagram also tells us whether a critical point is a peak, a trough or a saddle point. Consider, for instance, \( x = -3 \). To the left of \( x = -3 \) the derivative is negative and hence the function decreasing. To the right the derivative is positive and hence the function increasing. Therefore \( x = -3 \) is a trough.

We conclude that the function has peaks at \( x = -2 \) and \( x = 3 \). It has troughs at \( x = -3 \) and \( x = 2 \).

### 4.2 Inflection points

The second derivative is

\[y'' = f''(x) = (-x^4 + 13x^2 - 36)'
= -4x^3 + 26x\]
We have $y'' = -4x \left(x^2 - \frac{13}{2}\right)$. So, $x = 0$ is clearly a candidate for an inflection point. Moreover, $x = \pm \sqrt{\frac{13}{2}} \approx \pm 2.55$ are the solutions to $x^2 - \frac{13}{2} = 0$. These solutions are also candidates for inflection points. We obtain the factorization

$$y'' = f''(x) = -4x \left(x + \sqrt{\frac{13}{2}}\right) \left(x - \sqrt{\frac{13}{2}}\right)$$

and hence the sign diagram

<table>
<thead>
<tr>
<th></th>
<th>$x &lt; -\sqrt{\frac{13}{2}}$</th>
<th>$x = -\sqrt{\frac{13}{2}}$</th>
<th>$-\sqrt{\frac{13}{2}} &lt; x &lt; 0$</th>
<th>$x = 0$</th>
<th>$0 &lt; x &lt; \sqrt{\frac{13}{2}}$</th>
<th>$x = \sqrt{\frac{13}{2}}$</th>
<th>$\sqrt{\frac{13}{2}} &lt; x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4x$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$x + \sqrt{\frac{13}{2}}$</td>
<td>$-$</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>$+$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$x - \sqrt{\frac{13}{2}}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>$+$</td>
<td>0</td>
<td>$-$</td>
<td>0</td>
<td>$+$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

We see that all candidates for inflection points are indeed candidates for inflection points because the second derivative changes sign. Moreover, the function is convex for $x < -\sqrt{\frac{13}{2}}$ and for $0 < x < \sqrt{\frac{13}{2}}$. It is concave for $-\sqrt{\frac{13}{2}} < x < 0$ and $\sqrt{\frac{13}{2}} < x$.

### 4.3 The final picture

The following “summary sign diagram” summarizes the main findings.

<table>
<thead>
<tr>
<th></th>
<th>$x$ :</th>
<th>$\ldots$</th>
<th>$-3$</th>
<th>$\ldots$</th>
<th>$-\sqrt{\frac{13}{2}}$</th>
<th>$\ldots$</th>
<th>$-2$</th>
<th>$\ldots$</th>
<th>$0$</th>
<th>$\ldots$</th>
<th>$2$</th>
<th>$\ldots$</th>
<th>$\sqrt{\frac{13}{2}}$</th>
<th>$\ldots$</th>
<th>$3$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
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<td>$+$</td>
<td>$+$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>$-$</td>
<td>$0$</td>
<td>$+$</td>
<td>$+$</td>
<td>$0$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$0$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$0$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$0$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$0$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$0$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

If you have not seen it before, this was quite some bit of analytic geometry. Without drawing any graphs we determined the main qualitative features of the graph of the function. We used some very basic differentiation and some slightly more sophisticated algebra (although, in the end it was just solving quadratic equations). You should take some time out to check how the information in the tables is reflected in the graphs below. You should also check that what we have said about peaks, inflection points etc. is visible in the graphs.
5 A rational function

This is Example 3.7 from the textbook

\[ y = f(x) = \frac{x}{(x + 1)^2} \]

1. The numerator has degree 1, the denominator degree 2. Hence the horizontal axis is an asymptote.

2. The denominator is always non-negative. Hence \( f(x) \) is positive or negative according to whether \( x \) is positive or negative.

3. The denominator is zero for \( x = -1 \). Hence the vertical line at this point is an asymptote. We have \( f(x) = \frac{1}{(x+1)^2} \times \frac{x}{1} \) where the first part is not defined at \( x = -1 \) and the second part is -1 when \( x = -1 \). Therefore the function behaves near \( x = -1 \) similar to the function \( y = -\frac{1}{x^2} \) near \( x = 0 \). Hence, as we approach \( x = -1 \) from the left or the right, the values of the function explode towards \(-\infty\).
4. Summarizing the information so far:

| $x$ | $-\infty$ | $x < -1$ | $x = -1$ | $-1 < x < 0$ | $x = 0$ | $0 < x$ | $\rightarrow +\infty$
|-----|---------|---------|---------|-------------|---------|---------|------------|
| $f(x)$ | $0$ | $-\rightarrow -\infty$ | $\times$ | $\rightarrow -\infty$ | $\times$ | $-\rightarrow 0$ | $+$ | $0$ | $+$ | $0$

5. The first derivative is (using the quotient rule and the general power rule)

$$y'(x) = \frac{(x)^{(x+1)^2} - x ((x+1)^2)^'}{((x+1)^2)^2} = \frac{(x+1)^2 - x (2 (x+1) (x+1))}{(x+1)^4}$$

$$= \frac{(x+1)^2 - 2x (x+1)}{x+1} = \frac{(x+1) - 2x}{x+1} = -\frac{x+1}{x+1}$$

which gives the sign diagram

| $x$ | $x < -1$ | $x = -1$ | $-1 < x < 1$ | $x = 1$ | $1 < x$
|-----|---------|---------|-------------|---------|---------|
| $-x + 1$ | $+$ | $+$ | $+$ | $0$ | $-$
| $(x+1)^3$ | $-$ | $0$ | $+$ | $+$ | $+$
| $f'(x)$ | $-$ | $\times$ | $+$ | $0$ | $-$

Hence the function is decreasing to the left of $-1$, increasing to the right until it reaches a peak at $x = 1$ and is then decreasing again.

6. The second derivative is

$$f''(x) = \frac{(-1) (x+1)^3 - (x+1) (3 (x+1)^2)}{(x+1)^6} = \frac{-x - 1 + 3x - 3}{(x+1)^4} = 2x - 4$$

which gives the sign diagram

| $x$ | $x < -1$ | $x = -1$ | $-1 < x < 2$ | $x = 2$ | $2 < x$
|-----|---------|---------|-------------|---------|---------|
| $2x - 4$ | $-$ | $-$ | $-$ | $0$ | $+$
| $(x+1)^4$ | $+$ | $0$ | $+$ | $+$ | $+$
| $f''(x)$ | $-$ | $\times$ | $-$ | $0$ | $+$

Instead of giving a summary sign diagram, let me describe verbally the information obtained by the above calculations:

Coming from $-\infty$ on the left the graph of the function is close to, but below the horizontal axis. As $x$ increases it remains below it and is decreasing and downward-bowed (concave) until it “explodes” towards $-\infty$ to the left of $-1$. To the right of $-1$ the function shoots up from $-\infty$ and is increasing and downward-bowed. It crosses the horizontal axis at $x = 0$, but remains increasing until it reaches a peak at $x = 1$. From there onward it is decreasing but it remains positive. There is an inflection point at $x = 2$ where the graph turns from being downward bowed to being upward bowed. From there onwards the graph approaches more and more the horizontal axis from above.
We can deduce all this before turning on the graphic calculator. Of course it is useful to draw the graph in order to see that we are correct:

Fig. 12: \( y(x) = \frac{x}{(x+1)^2} \)

**Exercise 2** Read carefully through example 3.8 of the textbook. Verify that the first and second derivative are correct. Unluckily the inflection point can only be found numerically in this example.

**Reading:** (Hoffmann and Bradley 2000), Chapter 3, Sections 1 - 3

**References**