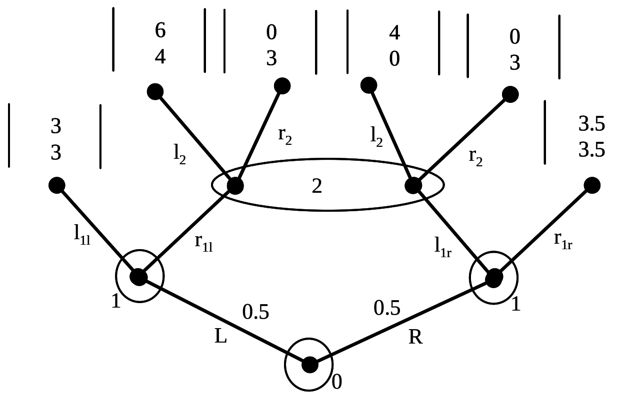
BEEM105 – GAME THEORY AND APPLICATIONS	Dieter Balkenborg Departments of Economics
Solving a simple signalling game	University of Exeter

1 Simplified game used in experiment



$$
\begin{aligned}
p &= \text{prob}\,(r_{1l}) \\
q &= \text{prob}\,(r_{1r}) \\
\rho &= \text{prob}\,(r_2) \\
\mu &= \text{prob}\,(\text{play is in right node} \mid \text{2's information set is reached})
\end{aligned}
$$

$$
\begin{aligned}
u_2(l|\mu) &> u_2(r|\mu) \\
4(1-\mu) &= 3 \Leftrightarrow \mu = 1/4
\end{aligned}
$$

For $p > 0$ or $q < 1$ the belief μ is determined via Bayesian updating as

$$
\mu = \frac{0.5(1-q)}{0.5p + 0.5(1-q)} = \frac{1-q}{p+1-q}
$$

Therefore player 2 should at his information set choose

$$
\begin{array}{rl}
\rho = 0 & \text{if} \quad \mu < 1/4 \\
\rho \in [0,1] & \text{if} \quad \mu = 1/4 \\
\rho = 1 & \text{if} \quad \mu > 1/4
\end{array}
$$

Given the probability ρ with which player 2 chooses right at his information set, player 1 should choose r at her left information set if

$$
\begin{aligned}
u_1(l_{1l}, q, \rho) &< u_1(r_{1l}, q, \rho) \\
3 &< 6(1-\rho) + 0 \times \rho \Leftrightarrow \rho < 1/2
\end{aligned}
$$

She should choose right at her right information set if

$$
\begin{aligned}
u_1(p, l_{1r}, \rho) &< u_1(p, l_{1l}, \rho) \\
4(1-\rho) &< 3.5 \Leftrightarrow \rho < 1/8
\end{aligned}
$$

Thus player 1's best responses are as follows:

$$
\begin{array}{cccc}
p = 1 & q = 0 & \text{for} & \rho < 1/8 \\
p = 1 & q \in [0,1] & \text{for} & \rho = 1/8 \\
p = 1 & q = 1 & \text{for} & 1/8 < \rho < 1/2 \\
p \in [0,1] & q = 1 & \text{for} & \rho = 1/2 \\
p = 0 & q = 1 & \text{for} & 1/2 < \rho
\end{array}
$$

We now try to fin a Nash equilibrium for the various values of ρ, assuming that ρ is the equilibrium probability with which player 2 chooses right. As the table shows, we must consider 5 cases.

1. $\rho = 0$. Player 1 plays (r_{1l}, l_{1r}) in a best reply. Player 2's information is always reached and $\mu = \frac{1}{1+1} = \frac{1}{2}$. Player 2 must choose r in a best reply, i.e. $\rho = 1$, given this belief. We cannot have a Nash equilibrium.

2. $0 < \rho < 1/8$. Player 1 plays (r_{1l}, l_{1r}) in a best reply. Player 2's information is always reached and $\mu = \frac{1}{1+1} = \frac{1}{2}$. Player 2 must choose r in a best reply, i.e. $\rho = 1$, given this belief. We cannot have a Nash equilibrium.

3. $\rho = 1/8$. Player 1 plays $(r_{1l}, (1-q)\, l_{1r} + q r_{1r})$ with $q \in [0,1]$ in a best reply. Player 2's information is always reached and $\mu = \frac{1-q}{1+1-q} = \frac{1-q}{2-q}$. Player 2 must be indifferent and hence $\mu = 1/4$. Thus

$$
\frac{1}{4} = \mu = \frac{1-q}{2-q} \Leftrightarrow q = \frac{2}{3}
$$

   We obtain the Nash equilibrium

$$
\left(\left(r_{1l}, \frac{1}{3} l_{1r} + \frac{2}{3} r_{1r}\right), \frac{7}{8} l_2 + \frac{1}{8} r_2\right)
$$

4. $1/8 < \rho < 1/2$. Player 1 plays (r_{l1}, r_{1r}) in a best reply. In a best response to that player 2 must choose l_2 with certainty, i.e. $\rho = 0$. thus there is no Nash equilibrium in this case.

5. $\rho = 1/2$. Player 1 plays $((1-p)\, l_{1l} + p r_{l1}, r_{1r})$ with $p \in [0,1]$ in a best reply. Player 2's conditional probability for being in the right node when his information set is reached is hence $\mu = 0/p = 0$ for $p > 0$. However, then he should play l_2 with certainty, contradicting $\rho = 1/2$. So we must have $p = 0$ and the information set is never reached. We obtain the Nash equilibrium

$$
\left((l_{1l}, r_{1r}), \frac{1}{2} l_2 + \frac{1}{2} r_2\right)
$$

This equilibrium can be supported as perfect (or sequential) Nash equilibrium with the belief $\mu = 1/2$. The approximating Nash equilibrium is

$$\left(\left(\left(1 - \varepsilon_{1l}\right) l_{1l} + \varepsilon_{1l} r_{1l}, \varepsilon_{1r} l_{1r} + \left(1 - \varepsilon_{1r}\right) r_{1r} \right), \frac{1}{2} l_2 + \frac{1}{2} r_2 \right)$$

in the perturbed game where every player trembles and chooses any of his strategies with a small positive probability. In particular, player 1 trembles such that she chooses r_{1l} by accident with probability ε_{1l} and l_{1r} with probability $\frac{2}{3}\varepsilon_{1r} = \frac{2}{3}\varepsilon_{1l}$. (The trembling probabilities are here included in the strategies.)

6. $1/2 < \rho < 1$. All strategy combinations

$$\left(\left(l_{1l}, r_{1r} \right), (1 - \rho) l_2 + \rho r_2 \right)$$

for $1/2 < \rho < 1$ are Nash equilibria as above. For $\rho < 1$ one needs $\varepsilon_{1r} = \frac{2}{3}\varepsilon_{1l}$ in a trembling hand approximation to induce the belief $\mu = 1/4$ and hence make player 2 indifferent between his choices. (However,

$$\left(\left(l_{1l}, r_{1r} \right), r_2 \right)$$

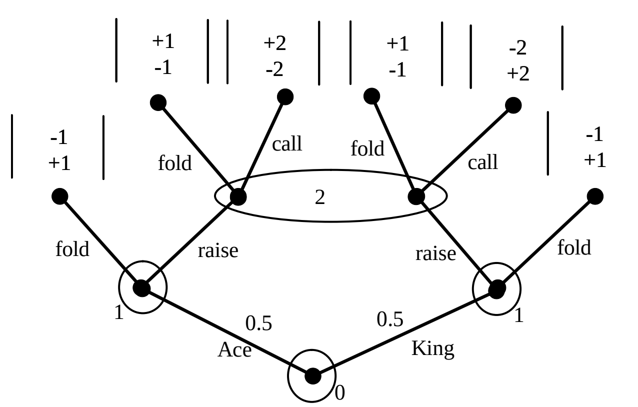
can be approximated by a much wider class of trembles (one only needs $\varepsilon_{1r} \geq \frac{2}{3}\varepsilon_{1l}$). In particular $\left(\left(l_{1l}, r_{1r} \right), r_2 \right)$ is the only equilibrium that can be approximated with uniform trembles (i.e. all choices are chooses by error with the same small probability).)

7. $\rho = 1$. The strategy combination

$$\left(\left(l_{1l}, r_{1r} \right), r_2 \right)$$

is a Nash equilibrium.

2 Simplified Poker



The following in a Nash equilibrium:

$$\begin{aligned} p &= \text{prob (player 1 raises with an ace)} = 1 \\ q &= \text{prob (player 1 raises with a king)} = 1/3 \\ r &= \text{prob (player 2 calls)} = 2/3 \end{aligned}$$

Indeed, if player 1 has an ace he should always raise, regardless of whether player 2 folds or not because both +1 and +2 are better than -1.

If player 1 has an ace he is better off raising if player 2 folds, but better off folding if player 2 calls. In equilibrium he expects player 2 to fold with probability 2/3. Therefore his expected payoff (conditional on having a king) if he raises is

$$\frac{1}{3} \times (+1) + \frac{2}{3} \times (-2) = -1$$

which is the same payoff as he gets when he folds. Therefore, given the behaviour of player 2, both raising and folding are optimal. Consequently, by always raising if he gets an ace and by raising with 1/3 probability and folding with 2/3 probability if he has a king, player 1 is playing optimal given the behavior of player 2.

Suppose now player 1 behaves as described. Then player 2's information set (consisting of two nodes) is reached with a total probability of

$$\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{3} = \frac{4}{6}.$$

The conditional probability of being in the left node is

$$\frac{1}{2} / \frac{4}{6} = \frac{3}{4}$$

This means: In equilibrium, if player 2 sees that player 1 raises, then there is a probability of $\frac{3}{4}$ that he has an ace and a $\frac{1}{4}$ probability that he has a king. If player 2 folds she gets a -1 in any case. If she calls she gets -2 with probability $\frac{3}{4}$ and $+2$ with probability $\frac{1}{4}$. Her expected payoff is

$$\frac{3}{4} \times (-2) + \frac{1}{4} \times (+2) = \frac{2-6}{4} = -1.$$

Hence she gets the same expected payoff with both her actions, which are hence both optimal. Therefore, also folding with probability 1/3 and calling with probability 2/3 is optimal.

In summary, both players react optimally to each other, as required for a Nash equilibrium.

Problem 1 *Show that the simplified Poker game has a unique Nash equilibrium. (Proceed as in the preceeding section. Player 2's mixed strategy set can be identified with the unit interval* $0 \leq \rho \leq 1$*. Determine first for what mixed strategies of player 2 player 1 is indifferent between his two actions at one of his two information sets. This gives you two points in the unit interval which separate it into three subintervals. Consider 7 cases: The* *two endpoint of the interval $\rho = 0$ and $\rho = 1$, the interior of the three subintervals and the two points in the middle. In each case, find player 1's best responses against the mixed strategy of player 2. (Only for the two points in the middle there will be multiple best responses.) For every best response of player 1 use Bayes' rule to calculate the induced belief of player 2 at her information set (this is any belief if the information set is never reached) and check whether she is playing a best reply. If yes, you have found a Nash equilibrium, otherwise not.)*