BEEM103 Mathematics for Economists

Constrained Optimization 1

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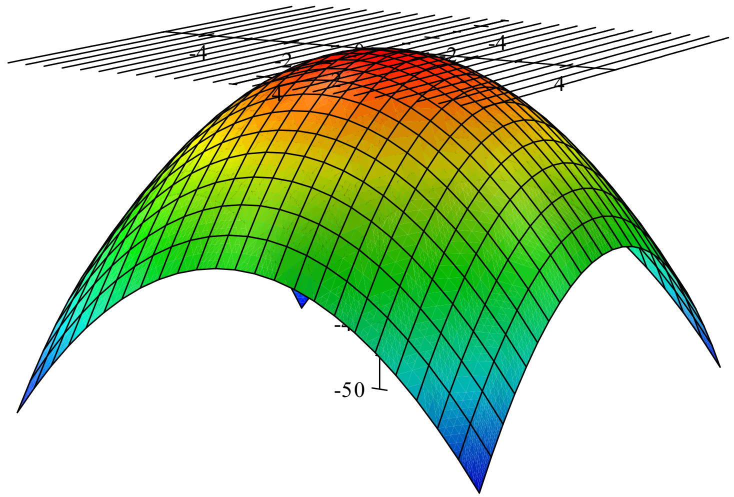
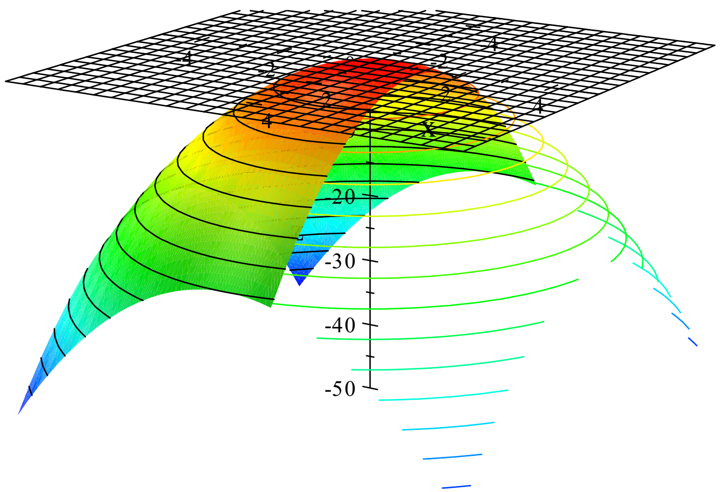
Week 3

Unconstrained Optimization



Constrained Optimization:

$y \geq 1$



Constrained Optimization

Examples:

1. A consumer maximizes his utility subject to his budget constraint.
2. A producer minimizes costs subject to the constraint that a certain amount is produced.
3. Moral hazard: An insurer tries to select an insurance contract that maximizes profits subject to the constraints that it is valuable to the consumer ("Participation Constraint") and that the consumer has an incentive to be careful ("Incentive Constraint"). Basic result: Full insurance is not optimal because it would make consumer act careless.

Constrained Optimization Problem

Objective: Find the (absolute) maximum of the function

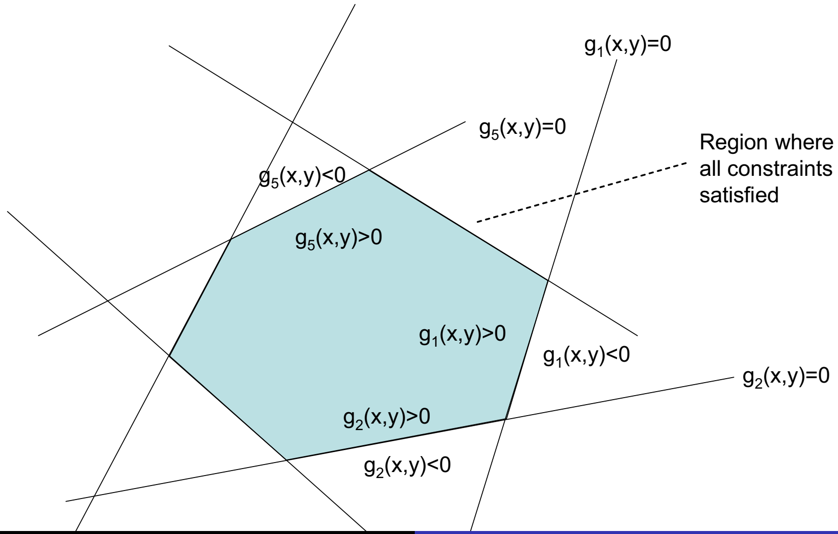
$$z = f(x, y)$$

subject to the inequality constraints

$$\begin{aligned} g_1(x,y) &\geq 0 \\ g_2(x,y) &\geq 0 \\ &\vdots \\ g_K(x,y) &\geq 0 \end{aligned}$$

Thus find pair (x^*, y^*) satisfying the constraints such that we have for all other pairs (x, y) satisfying the constraints: $f(x^*, y^*) \geq f(x, y)$.
$f(x, y)$ is called the "objective function" and I call
$g_1(x,y), \ldots, g_K(x,y)$ the "constraining functions" of the problem.

The constraints carve out a region of the plane.



Example 1: Consumer Optimization

A consumer wants to maximize his utility

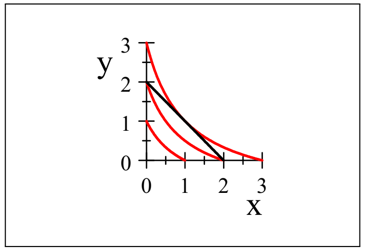
$$u(x,y) = (x+1)(y+1)$$

subject to his budget constraint

$$b - p_x x - p_y y \geq 0$$

and the non-negativity constraints

$$x \geq 0 \qquad y \geq 0$$



Example 2: Cost Minimization

A producer with production function $Q(K,L) = K^{\frac{1}{6}} L^{\frac{1}{2}}$ in a perfectly competitive market wants to *minimize* costs subject to producing at least Q_0 units.
Maximize

$$-(rK + wL)$$

subject to

$$\begin{aligned} Q(K,L) - Q_0 &\geq 0 \\ K &\geq 0 \\ L &\geq 0 \end{aligned}$$

Example 3: Shortest Route

A swimmer who is currently at the coordinates (a, b) wants to swim along the shortest route to the square island with corner points $(-1, 1)$, $(1, -1)$, $(-1, 1)$, $(1, 1)$.
Instead of minimizing the distance we can maximize the negative of the square of the distance

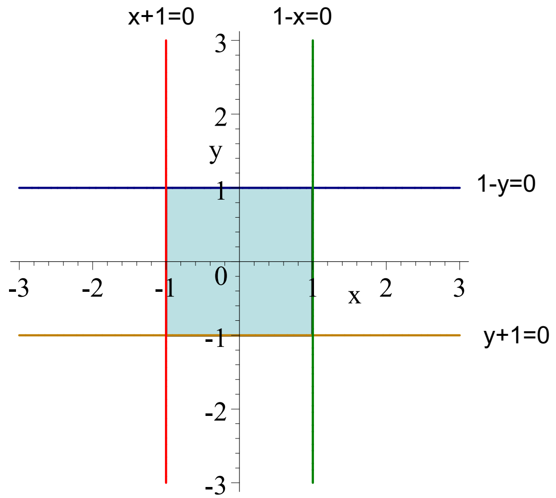
$$-(x-a)^2 - (y-b)^2$$

subject to the constraints

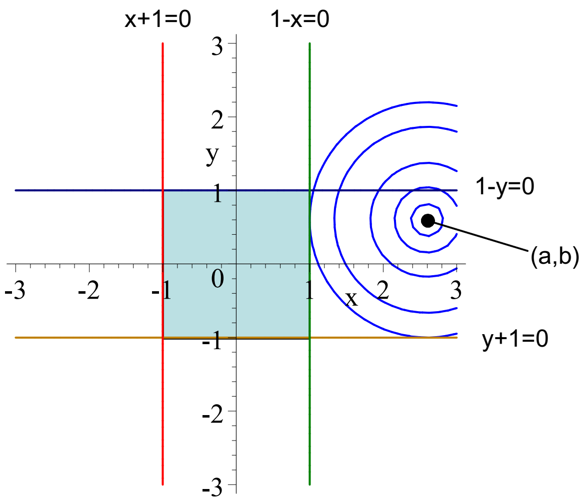
$$\begin{aligned} x+1 &\geq 0 \\ 1-x &\geq 0 \\ y+1 &\geq 0 \\ 1-y &\geq 0 \end{aligned}$$

(x, y) is a point in the square carved out by the solutions to the four inequalities.

The Island

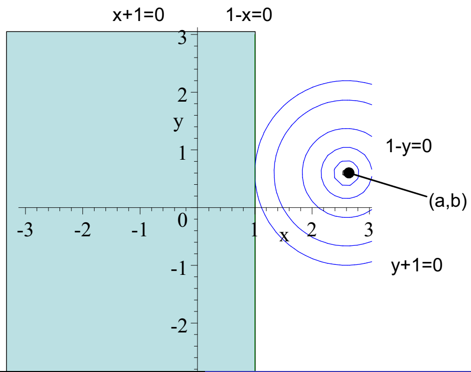


The Swimmer



Binding constraints

A constraint is *binding at the optimum* if it holds with equality in the optimum. In the above picture only one of the four the constraints is binding. All non-binding constraints can be ignored. If they are left out the optimum does not change.



The Lagrangian Approach

The Lagrangian approach transfers a constrained optimization problem into

1. an unconstrained optimization problem and
2. a pricing problem.

The new function to be optimized is called the *Lagrangian*. For each constraint a shadow price is introduced, called a *Lagrange multiplier*.
In the new unconstrained optimization problem a constraint can be violated, but only at a cost.
The pricing problem is to find shadow prices for the constraints such that the solutions to the new and the original optimization problem are identical.

The Lagrangian Approach

The Lagrangian:

$$\mathcal{L}(x,y) = f(x,y) + \lambda_1 g_1(x,y) + \lambda_2 g_2(x,y) + \cdots + \lambda_K g_K(x,y)$$

Theorem

Suppose we are given numbers $\lambda_1, \lambda_2, \ldots, \lambda_K \geq 0$ *and a pair of numbers* (x^*, y^*) *such that*

1. $\lambda_1, \lambda_2, \ldots, \lambda_K \geq 0$, *i.e. Lagrange mutlipliers are non-negative,*
2. (x^*, y^*) *satisfies all the constraints, i.e.,* $g_k(x^*, y^*) \geq 0$ *for all* $1 \leq k \leq K$.
3. (x^*, y^*) *is an unconstrained maximum of the Lagrangian* $\mathcal{L}(x,y)$.
4. *The complementarity conditions*

$$\lambda_k g_k(x^*, y^*) = 0$$

are satisfied, i.e., either the k-Lagrange multiplier is zero or the k-th constraint binds for $1 \leq k \leq K$.

Then (x^*, y^*) *is a maximum for the constrained maximization*

The Lagrangian:

$$\mathcal{L}(x,y) = f(x,y) + \lambda_1 g_1(x,y) + \lambda_2 g_2(x,y) + \cdots + \lambda_K g_K(x,y)$$

Proof.

Because of the complementarity conditions we have $\mathcal{L}(x^*, y^*) = f(x^*, y^*)$. We have $\mathcal{L}(x,y) \leq \mathcal{L}(x^*, y^*)$ for any (x,y). If (x,y) satisfies all constraints then $\lambda_k g_k(x,y) \geq 0$ for each constraint since the λ_k are non-negative. Hence $f(x,y) \leq \mathcal{L}(x,y)$ and therefore $f(x,y) \leq f(x^*, y^*)$ for any point (x,y) satisfying the constraints. □

The Method

1. Make an informed guess about which constraints are binding at the optimum. (Suppose there are k^* such constraints.)
2. Set the Lagrange multipliers for all other constraints zero, i.e. ignore these constraints.
3. Solve the two first -order conditions $\frac{\partial \mathcal{L}}{\partial x} = 0$, $\frac{\partial \mathcal{L}}{\partial y} = 0$ together with the conditions that the k^* constraints are binding. (Notice that we have $2 + k^*$ constraints and equations, namely x, y and k^* Lagrange multipliers.)
4. Check whether the solution is indeed an unconstrained optimum of the Lagrangian. (May be difficult.)
5. Check that the Lagrange multipliers are all non-negative and that the solution (x^*, y^*) satisfies all constraints.
6. If 4. and 5. are violated, start again at 1. with a new guess.

The Swimmer's Problem

The Lagrangian is

$$\begin{aligned}\mathcal{L}(x,y) &= -(x-a)^2 - (y-b)^2 \\ &\quad + \lambda_1 (x+1) + \lambda_2 (1-x) + \lambda_3 (y+1) + \lambda_4 (1-y)\end{aligned}$$

FOC:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= -2(x-a) + \lambda_1 - \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= -2(y-b) + \lambda_3 - \lambda_4 = 0\end{aligned}$$

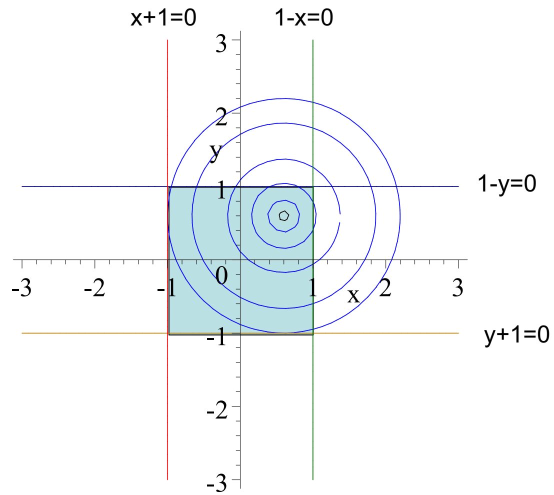
Case 1:

Suppose $-1 < x^* < 1$, $-1 < y^* < 1$.
None of the four constraints is "binding"
In this case the optimum has to be a stationary point of the objective function. This gives the conditions

$$\begin{aligned}\frac{\partial f}{\partial x} &= -2(x-a) = 0 \\ \frac{\partial f}{\partial y} &= -2(y-b) = 0\end{aligned}$$

UNIQUE SOLUTION: $(x^*, y^*) = (a, b)$. Must have: $-1 \leq a, b \leq 1$ for this to be the optimum.

Case 1



Case 2

Optimum is on the upper side of the square, but not a cornerpoint, i.e. the ONLY binding constraint is $y \leq 1$. All Lagrange multipliers except λ_4 must be zero. So the Lagrangian is

$$\mathcal{L}(x, y) = f(x, y) + \lambda_4 (1 - y)$$

FOC:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= -2(x - a) = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= -2(y - b) - \lambda_4 = 0 \end{aligned}$$

Add that the constraint $y = 1$ is binding, i.e.

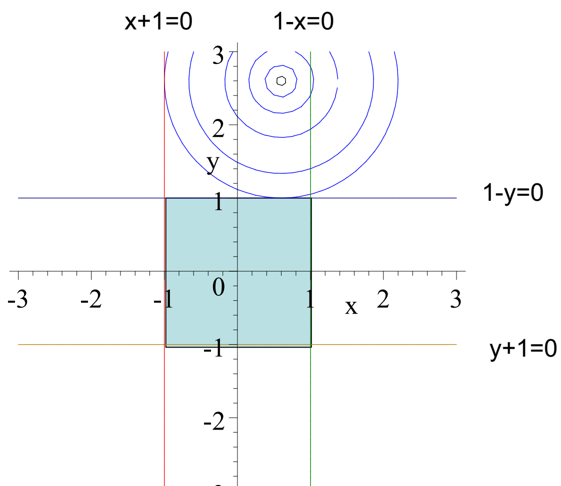
$$y = 1.$$

unique solution $x^* = a, y^* = 1, \lambda_4 = -2(1 - b)$.

Case 2

For our candidate $(x^*, y^*) = (a, 1)$ to satisfy all constraints we must have $-1 \leq a \leq 1$. In addition, $\lambda_4 = 2(b - 1)$ must be nonnegative. True only if $b - 1 \geq 0$ or $b \geq 1$.
Cases 3,4,5: Optimum is on a different side.
Solved symmetrically

Case 2



Case 6

Suppose optimum is cornerpoint $(1, 1)$. Then constraints $x \leq 1$ and $y \leq 1$ are binding while $x \geq -1$ and $y \geq -1$ are not. Complementarity conditions imply $\lambda_1 = \lambda_3 = 0$. Lagrangian is

$$\mathcal{L}(x, y) = f(x, y) - \lambda_2 (x - 1) - \lambda_4 (y - 1)$$

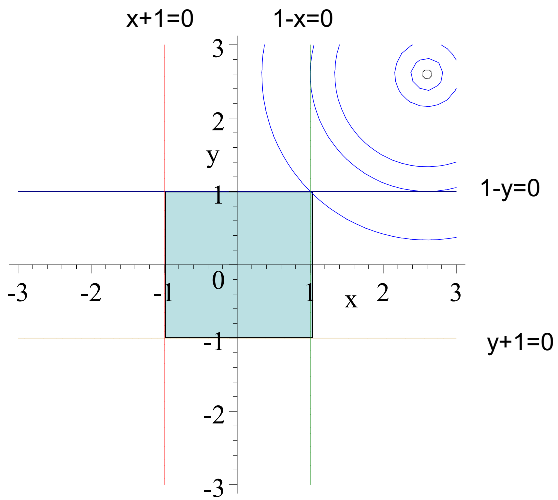
FOC:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= -2(x - a) - \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= -2(y - b) - \lambda_4 = 0 \end{aligned}$$

cornerpoint:

$$\begin{aligned} x &= 1 \\ y &= 1. \end{aligned}$$

Case 6



Case 6

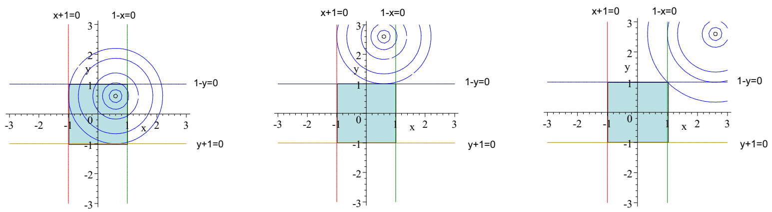
unique solution: $(x^*, y^*) = (1, 1)$, $\lambda_2 = -2(1-a)$, $\lambda_4 = -2(1-b)$. The Lagrange multipliers are non-negative if

$$a \geq 1 \quad \text{and} \quad b \geq 1.$$

Cases 7, 8, 9: Optimum is another corner point.
The following table describes for each pair (a, b) what the optimum (x^*, y^*) is

	$b \leq -1$	$-1 < b < 1$	$1 \leq b$
$a \leq -1$	$(-1, -1)$	$(-1, b)$	$(-1, 1)$
$-1 < a < 1$	$(a, -1)$	(a, b)	$(a, 1)$
$1 \leq a$	$(1, -1)$	$(1, b)$	$(1, 1)$

The Lagrangian Approach



- Is the optimum inside, on an edge or at a cornerpoint of the region carved out by the inequality constraints?
- I.e. which constraints hold with equality in the optimum?
- A constraint is "binding" at the optimum, if it holds there with equality.
- A constraint is "binding" if it has bite and restrains the optimum.

- A constraint which is not binding can be ignored without changing the optimum.

The Lagrangian:

$$\mathcal{L}(x, y) = f(x, y) + \lambda_1 g_1(x, y) + \lambda_2 g_2(x, y) + \cdots + \lambda_K g_K(x, y)$$

Theorem

Suppose we are given numbers $\lambda_1, \lambda_2, \ldots, \lambda_K \geq 0$ and a pair of numbers (x^*, y^*) such that

1. *$\lambda_1, \lambda_2, \ldots, \lambda_K \geq 0$, i.e. Lagrange mutlipliers are non-negative,*
2. *(x^*, y^*) satisfies all the constraints, i.e., $g_k(x^*, y^*) \geq 0$ for all $1 \leq k \leq K$.*
3. *(x^*, y^*) is an unconstrained maximum of the Lagrangian $\mathcal{L}(x, y)$.*
4. *The complementarity conditions*

$$\lambda_k g_k(x^*, y^*) = 0$$

are satisfied, i.e., either the k-Lagrange multiplier is zero or the k-th constraint binds for $1 \leq k \leq K$.

Then (x^*, y^*) is a maximum for the constrained maximization

- The Lagrangian approach does not immediately tell you which constraints are binding in the optimum, you will have to start with an informed guess using all problem-specific information.
- You write down the Lagrangian assuming that only certain constraints bind. For the others the Lagrange multipliers and the corresponding terms $\lambda_k g_k(x, y)$ are zero. (Often these are already ignored in the beginning.)
- You solve the simultaneous system of equations consisting of the FOC and the equations $g_k(x, y) = 0$ for the constraints assumed to be binding.
- You check that the solution satisfies the other constraints (which are not assumed to be binding) and that the Lagrange multiplicators are non-negative.
- You check that the solution found is an unconstrained optimum of the Lagrangian.
- If all this holds, you have found the optimum. Otherwise, try again.

Notice:

- If only one constraint binds, say g_1, the FOC become

$$\frac{\partial f}{\partial x} = -\lambda_1 \frac{\partial g_1}{\partial x}$$
$$\frac{\partial f}{\partial y} = -\lambda_1 \frac{\partial g_1}{\partial y}$$

and must hold together with $g_1(x, y) = 0$. This simplifies to the two equations

$$\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} = \frac{\partial g_1}{\partial x} \Big/ \frac{\partial g_1}{\partial y} \qquad g_1(x, y) = 0$$

- $$\frac{\partial \mathcal{L}}{\partial \lambda_k} = g_k(x, y)$$

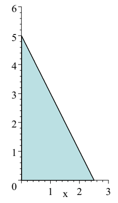
Maximize $f(x, y)$ subject to the constraints $h_k(x, y) \leq 0$ $(k = 1, \cdots, K)$. Equivalently $g_k(x, y) \geq 0$ with $g_k(x, y) = -h_k(x, y)$.
The Lagrangian:

$$\mathcal{L}(x, y) = f(x, y) + \lambda_1 g_1(x, y) + \lambda_2 g_2(x, y) + \cdots + \lambda_K g_K(x, y)$$
$$= f(x, y) - \lambda_1 h_1(x, y) - \lambda_2 h_2(x, y) - \cdots - \lambda_K h_K(x, y)$$

- Textbooks do not "converge" on one approach.
- For the algebra all this does not matter, at the worst you may get negative Lagrange multipliers.
- To avoid further confusion I rewrite minimization problems as maximization problems.

Utility maximization

Maximize $u(x,y) = (x+1)(y+1)$, subject to the budget constraint $p_x x + p_y y \leq b$ and the non-negativity constraints $x, y \geq 0$.
Where can the consumer optimum be in the budget set carved out by the budget- and the non-negativity constraints?



In principle there are 7 different possibilities to consider: It could be in the interior of the triangle, on one of the three sides or it could be one of the three corner points.

Utility maximization

$u(x,y) = (x+1)(y+1)$ is "monotonic", more of each commodity is better.

Lemma

If the utility function is monotonic then the budget constraint must be binding in the consumer optimum, i.e. the optimum is on the budget line.

Proof.

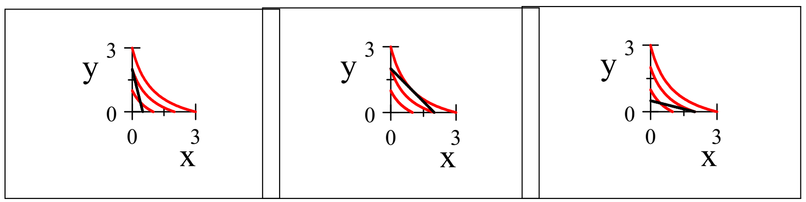
If (x,y) is a consumption bundle which costs less than the budget then one can, for instance, slightly increase the consumption of x without violating the budget constraint. The first factor in $(x+1)(y+1)$ increases and hence, since $y+1>0$ the whole product. Utility goes up and so (x,y) cannot be the optimum. □

This argument holds generally for all *monotonic preferences*.

Utility maximization

- This reduces the search to three possibilities: The budget line and its two corner points. (Different: "bads", satiated consumers.)

Three possibilities:



Optimum (0,2) Optimum (1,1) Optimum (2,0)

Utility maximization

- For a utility function like $u(x,y) = xy$ *one can say even more*, namely that consumption of both commodities must be strictly positive in optimum. (Utility is zero when $x = 0$ or $y = 0$ while a strictly positive utility can be achieved with a strictly positive budget.) Thus *only* the budget constraint can be binding.

Utility maximization - The Lagrangian

$$\begin{aligned}\mathcal{L}(x,y) &= u(x,y) + \lambda_1 (b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y \\ &= (x+1)(y+1) + \lambda_1 (b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y\end{aligned}$$

where $p_x, p_y, b > 0$
FOC:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial u}{\partial x} - \lambda_1 p_x + \lambda_2 = (y+1) - \lambda_1 p_x + \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial u}{\partial y} - \lambda_1 p_y + \lambda_3 = (x+1) - \lambda_1 p_y + \lambda_3 = 0\end{aligned}$$

Case 1: Only the budget equation binds

Hence $\lambda_2 = \lambda_3 = 0$ by the complementarity condition. We get the FOC

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial u}{\partial x} - \lambda_1 p_x = (y+1) - \lambda_1 p_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial u}{\partial y} - \lambda_1 p_y = (x+1) - \lambda_1 p_y = 0\end{aligned}$$

and the budget constraint must hold with equality

$$p_x x + p_y y = b.$$

Notice that if x and y are postive, then the FOC imply $\lambda_1 > 0$.

Case 1: Only the budget equation binds

When only one constraint binds, it is easy to eliminate the Lagrange multiplier:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial u}{\partial x} - \lambda_1 p_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial u}{\partial y} - \lambda_1 p_y = 0\end{aligned}$$

implies

$$\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{p_x}{p_y}$$

Together with

$$p_x x + p_y y = b.$$

we must then solve a system of two equations with two unknowns.

Case 1: Only the budget equation binds

The FOC imply

$$\frac{\partial u}{\partial x} = \lambda_1 p_x \qquad \frac{\partial u}{\partial y} = \lambda_1 p_y$$

Division of the two left hand sides and the two right hand sides yields

$$\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{p_x}{p_y}$$

Thus the marginal rate of substitution (see previous lectures) must equal the price ratio or, in other words, in the consumer optimum the indifference curve is tangential to the budget line.

Case 1: Only the budget equation binds

In our particular example this yields

$$\frac{y+1}{x+1} = \frac{p_x}{p_y} \qquad y = \frac{p_x}{p_y}(x+1) - 1$$

Substitution into the budget equation yields.

$$\begin{aligned} b &= p_x x + p_y y = p_x x + p_y \left(\frac{p_x}{p_y}(x+1) - 1 \right) \\ b &= 2p_x x + p_x - p_y \end{aligned}$$

and so

$$x^* = \frac{b - p_x + p_y}{2p_x} \qquad y^* = \frac{b + p_x - p_y}{2p_y}$$

where the last formula holds because

$$y = \frac{p_x}{p_y}(x+1) - 1 = \frac{p_x}{p_y}\frac{b - p_x + p_y}{2p_x} + \frac{2p_x}{2p_y} - \frac{2p_y}{2p_y} = \frac{b + p_x - p_y}{2p_y}$$

Case 1: Only the budget equation binds

$$x^* = \frac{b - p_x + p_y}{2p_x} \qquad y^* = \frac{b + p_x - p_y}{2p_y}$$

can only be the solution when both numbers are non-negative. This requires

$$\begin{aligned} b - p_x + p_y &\geq 0 \qquad b + p_x - p_y \geq 0 \\ b &\geq p_x - p_y \qquad b \geq -(p_x - p_y) \\ b &\geq |p_x - p_y| \end{aligned}$$

where $||$ denotes the "absolute value"

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Intuitively, the price difference cannot be too large in comparison to the budget.

Case 1: Only the budget equation binds

To summarize, provided $b \geq |p_x - p_y|$ the Lagrangian approach yields a positive Lagrange multiplier λ_1 (see the argument further above) and the non-negative solution

$$x^* = \frac{b - p_x + p_y}{2p_x} \qquad y^* = \frac{b + p_x - p_y}{2p_y}$$

Provided we can show that this solution is indeed an unconstrained maximum of the Lagrangian (and not a minimum etc.) it is the solution to our constrained optimization problem.

Case 2: The budget equation binds and x*=0

So one of the non-negativity constraint is binding. Thus the first order conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial u}{\partial x} - \lambda_1 p_x + \lambda_2 = (y+1) - \lambda_1 p_x + \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial u}{\partial y} - \lambda_1 p_y + \lambda_3 = (x+1) - \lambda_1 p_y + \lambda_3 = 0 \end{aligned}$$

must holds together with the budget equation

$$p_x x + p_y y = b$$

and

$$x^* = 0$$

The budget equation simplifies to $p_y y = b$ and our only solution candidate is

$$x^* = 0 \qquad y^* = b/p_y$$

Case 2: The budget equation binds and x*=0

Because $y^* > 0$ the non-negativity constraint $y = 0$ does not bind and therefore $\lambda_3 = 0$ by the complementarity conditions. The FOC simplify to

$$\begin{aligned} b/p_y + 1 - \lambda_1 p_x + \lambda_2 &= 0 \\ 1 - \lambda_1 p_y &= 0 \end{aligned}$$

So $\lambda_1 = 1/p_y > 0$ and the first FOC yields

$$\begin{aligned} b/p_y + 1 - \frac{p_x}{p_y} + \lambda_2 &= -b/p_y - 1 + \frac{p_x}{p_y} \\ \frac{b + p_y - p_x}{p_y} + \lambda_2 &= 0 \\ \lambda_2 &= \frac{p_x - p_y - b}{p_y} \end{aligned}$$

For λ_2 to be non-negative we need that the price difference $p_x - p_y$ is bigger than the budget.

Case 3: The budget equation binds and y*=0

This case is handled completely symmetrically to case 2.

Summary

Apart form showing that we have indeed found unconstrained optima of the Lagrangian we get the following result

Theorem

- *When the price of x is very high, namely when $p_x \geq p_y + b$ the consumer only wants to buy y and so $x^* = 0$, $y^* = b/p_y$.*
- *When the price of y is very high, namely when $p_y \geq p_x + b$ the consumer only wants to buy y and so $x^* = b/p_x$, $y^* = 0$.*
- *In all other cases the consumer wants to buy of both commodities the amounts*

$$x^* = \frac{b - p_x + p_y}{2p_x} \qquad y^* = \frac{b + p_x - p_y}{2p_y}$$