A function

\[ z = f(x, y) \]

or simply

\[ z(x, y) \]

*in two independent variables with one dependent variable* assigns to each pair \((x, y)\) of (decimal) numbers from a certain domain \(D\) in the two-dimensional plane a number \(z = f(x, y)\). \(x\) and \(y\) are hereby the *independent variables*; \(z\) is the *dependent variable*. 
Example: The cubic polynomial

The graph of $f$ is the surface in 3-dimensional space consisting of all points $(x, y, f(x, y))$ with $(x, y)$ in $D$.

$$z = f(x, y) = x^3 - 3x^2 - y^2$$

Exercise: Evaluate $z = f(2, 1)$, $z = f(3, 0)$, $z = f(4, -4)$, $z = f(4, 4)$.

Example: production function

$$Q = \sqrt{K}\sqrt{L} = K^{\frac{1}{2}}L^{\frac{1}{2}}$$

capital $K \geq 0$, labour $L \geq 0$, output $Q \geq 0$
Outline

1. Functions in two variables
   - Example: The cubic polynomial
   - Example: production function
   - Example: profit function
   - Level Curves
   - Isoquants

2. Partial derivatives
   - A Basic Example
   - Notation
   - A Second Example
   - The Marginal Products of Labour and Capital

3. Optimization
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   - Example 3: Price Discrimination

4. Unconstrained Optimization

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Example: profit function

Assume that the firm is a price taker in the product market and in both factor markets.

- $P$ is the price of output
- $r$ the interest rate (= the price of capital)
- $w$ the wage rate (= the price of labour)
Example: profit function

Assume that the firm is a price taker in the product market and in both factor markets.

- $P$ is the price of output
- $r$ the interest rate (= the price of capital)
- $w$ the wage rate (= the price of labour)
- total profit of this firm:

$$\Pi(K, L) = TR - TC = PQ - rK - wL = PK^{\frac{1}{2}}L^{\frac{3}{2}} - rK - wL$$

- $P = 12$, $r = 1$, $w = 3$:

$$\Pi(K, L) = 12K^{\frac{1}{2}}L^{\frac{3}{2}} - K - 3L$$

Profits is maximized at $K = L = 8$. 

- $P = 12$, $r = 1$, $w = 3$:

$$\Pi(K, L) = PK^{\frac{1}{2}}L^{\frac{3}{2}} - rK - wL = 12K^{\frac{1}{2}}L^{\frac{3}{2}} - K - 3L$$
The level curve of the function \( z = f(x,y) \) for the level \( c \) is the solution set to the equation

\[
f(x,y) = c
\]

where \( c \) is a given constant.

Geometrically, a level curve is obtained by intersecting the graph of \( f \) with a horizontal plane \( z = c \) and then projecting into the \((x,y)\)-plane. This is illustrated on the next page for the cubic polynomial discussed above:

In the case of a production function the level curves are called isoquants. An isoquant shows for a given output level capital-labour combinations which yield the same output.
Finally, the linear function
\[ z = 3x + 4y \]
has the graph and the level curves:

The level curves of a linear function form a family of parallel lines:
\[ c = 3x + 4y \]
\[ 4y = c - 3x \]
slope \(-\frac{3}{4}\), variable intercept \( \frac{c}{4} \).

**Remark:** The exercises illustrate the following general principle: If \( h(z) \) is an increasing (or decreasing) function in one variable, then the composite function \( h(f(x, y)) \) has the same level curves as the given function \( f(x, y) \). However, they correspond to different levels.

**Exercise:** Describe the isoquant of the production function
\[ Q = KL \]
for the quantity \( Q = 4 \).

**Exercise:** Describe the isoquant of the production function
\[ Q = \sqrt{KL} \]
for the quantity \( Q = 2 \).

**Objectives for the week**

- Functions in two independent variables.

The lecture should enable you for instance to calculate the marginal product of labour.
Objectives for the week

- Functions in two independent variables.
- Level curves $\leftrightarrow$ indifference curves or isoquants

The lecture should enable you for instance to calculate the marginal product of labour.

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5. Second order conditions

Partial Derivatives: A Basic Example

Exercise: What is the derivative of
\[ z(x) = a^3 x^2 \]
with respect to $x$ when $a$ is a given constant?

Exercise: What is the derivative of
\[ z(y) = y^3 b^2 \]
with respect to $y$ when $b$ is a given constant?
Consider function \( z = f(x, y) \). Fix \( y = y_0 \), vary only \( x \):
\[
z = F(x) = f(x, y_0).
\]

- The derivative of this function \( F(x) \) is called the partial derivative of \( f \) with respect to \( x \) and denoted by
\[
\frac{\partial z}{\partial x} \bigg|_{y=y_0} = \frac{dF}{dx}.
\]

**Notation:** “\( \partial \)” = “delta”, “\( \delta \)” = “delta”, “\( d \)” = “del”

It suffices to *think* of \( y \) and all expressions containing only \( y \) as exogenously fixed constants. We can then use the familiar rules for differentiating functions in one variable in order to obtain \( \frac{\partial z}{\partial x} \).
Other common notations for partial derivatives are \( \frac{\partial f}{\partial x} \), \( \frac{\partial f}{\partial y} \) or \( f_x, f_y \) or \( f'_x, f'_y \).

Consider again function \( z = f(x, y) \). Fix \( y = y_0 \), vary only \( x \):
\[
z(x, y_0) = F(x) = f(x, y_0)
\]
Then evaluate at \( x = x_0 \)

Example: Let
\[
z(x, y) = y^3x^2
\]
Then
\[
\frac{\partial z}{\partial x} = y^32x = 2y^3x
\]
\[
\frac{\partial z}{\partial y} = 3y^2x^2
\]
Partial derivatives: A second example

Example: Let

\[ z = x^3 + x^2 y^2 + y^4. \]

Setting e.g. \( y = y_0 = 1 \) we obtain \( z = x^3 + x^2 + 1 \) and hence

\[ \frac{\partial z}{\partial x} |_{y=1} = 3x^2 + 2x \]

Evaluating at \( x = x_0 = 1 \) we then obtain:

\[ \frac{\partial z}{\partial x} |_{x=1,y=1} = 5 \]

For fixed, but arbitrary, \( y \) we obtain

\[ \frac{\partial z}{\partial x} = 3x^2 + 2xy^2 \]

as follows: We can differentiate the sum \( x^3 + x^2 y^2 + y^4 \) with respect to \( x \) term-by-term. Differentiating \( x^3 \) yields \( 3x^2 \), differentiating \( x^2 y^2 \) yields \( 2xy^2 \) because we think now of \( y^2 \) as a constant and \( \frac{d}{dx} (ax^2) = 2ax \) holds for any constant \( a \). Finally, the derivative of any constant term is zero, so the derivative of \( y^4 \) with respect to \( x \) is zero.

Similarly considering \( x \) as fixed and \( y \) variable we obtain

\[ \frac{\partial z}{\partial y} = 2x^2 y + 4y^3 \]
The Marginal Products of Labour and Capital

**Example:** The partial derivatives $\frac{\partial q}{\partial K}$ and $\frac{\partial q}{\partial L}$ of a production function $q = f(K, L)$ are called the marginal product of capital and (respectively) labour. They describe approximately by how much output increases if the input of capital (respectively labour) is increased by a small unit.

Fix $K = 64$, then $q = K^{\frac{1}{2}}L^{\frac{3}{2}} = 2L^{\frac{3}{2}}$ which has the graph

![Graph of $q = f(K, L)$](image)

**Diminishing productivity of labour:**

The more labour is used, the less is the increase in output when one more unit of labour is employed. Algebraically:

$$\frac{\partial q}{\partial L} = \frac{1}{2} K^{\frac{1}{2}} L^{-\frac{3}{2}} = \frac{1}{2} \frac{\sqrt{K}}{\sqrt{L}} > 0,$$

$$\frac{\partial^2 q}{\partial L^2} = \frac{\partial}{\partial L} \left( \frac{\partial q}{\partial L} \right) = -\frac{1}{4} K^{\frac{1}{2}} L^{-\frac{5}{2}} = -\frac{1}{4} \frac{\sqrt{K}}{\sqrt{L}^3} < 0.$$

**Exercise:** Find the partial derivatives of

$$z = (x^2 + 2x) (y^3 - y^2) + 10x + 3y$$

---

This graph is obtained from the graph of the function in two variables by intersecting the latter with a vertical plane parallel to $L-q$-axes.

![Graph of $z = f(x, y)$](image)

The partial derivatives $\frac{\partial q}{\partial K}$ and $\frac{\partial q}{\partial L}$ describe geometrically the slope of the function in the $K$- and, respectively, the $L$- direction.

---

"Since the fabric of the universe is most perfect, and is the work of a most perfect creator, nothing whatsoever takes place in the universe in which some form of maximum or minimum does not appear."

Leonhard Euler, 1744
Subject: Optimization of multivariate functions

Two basic types of problems:

- unconstrained (e.g. profit maximization)
- constrained
Objective: Find (absolute) maximum of function 

\[ z = f(x, y) \]

i.e., find a pair \((x^*, y^*)\) such that 

\[ f(x^*, y^*) \geq f(x, y) \]

holds for all pairs \((x, y)\).
Hereby both pairs of numbers \((x^*, y^*)\) and \((x, y)\) must be in the domain of the function.
For an absolute minimum require 

\[ f(x^*, y^*) \leq f(x, y) \]

\( f(x, y) \) is called the “objective function”.

The function

\[ z = f(x, y) = -(x - 2)^2 - (y - 3)^2 \]

has a maximum at \((x^*, y^*) = (2, 3)\).
First order conditions

The following must hold: freeze the variable $y$ at the optimal value $y^*$, vary only $x$ then the function in one variable

$$F(x) = f(x, y^*)$$

must have maximum at $x^*$: $\frac{dF}{dx}(x^*) = 0$. Thus we obtain the first order conditions

$$\frac{\partial z}{\partial x}|_{x=x^*, y=y^*} = 0 \quad \frac{\partial z}{\partial y}|_{x=x^*, y=y^*} = 0$$

Example 1

$$z = f(x, y) = - (x - 2)^2 - (y - 3)^2$$

$$\frac{\partial z}{\partial x} = -2(x - 2) \times (+1) = 0 \quad \frac{\partial z}{\partial y} = -2(y - 3) \times (+1) = 0$$

$x^* = 2 \quad y^* = 3$

The maximum (at least the only critical or stationary point) is at $(x^*, y^*) = (2, 3)$
Maximizing profits

Production function $Q(K, L)$.
- $r$ interest rate
- $w$ wage rate
- $P$ price of output
- profit $\Pi(K, L) = PQ(K, L) - rK - wL$.

FOC for profit maximum:

\[
\frac{\partial \Pi}{\partial K} = P \frac{\partial Q}{\partial K} - r = 0 \quad (1)
\]
\[
\frac{\partial \Pi}{\partial L} = P \frac{\partial Q}{\partial L} - w = 0 \quad (2)
\]

Rewrite FOC as

\[
\frac{\partial Q}{\partial K} = \frac{r}{P} \quad (3)
\]
\[
\frac{\partial Q}{\partial L} = \frac{w}{P} \quad (4)
\]

Division yields:

\[
MRS = -\frac{dL}{dK} = \frac{\frac{\partial Q}{\partial K}}{\frac{\partial Q}{\partial L}} = \frac{r}{w} \quad (5)
\]

The marginal rate of substitution must equal the ratio of the input prices!

Intuition: Suppose $P \frac{\partial Q}{\partial K} - r > 0$. By using one unit of capital more the firm could produce $\frac{\partial Q}{\partial K}$ units of output more. The revenue would increase by $P \frac{\partial Q}{\partial K}$, the cost by $r$ and so profit would increase. Thus we cannot have a profit optimum. If $P \frac{\partial Q}{\partial K} - r < 0$ it would symmetrically pay to reduce capital input. Hence $P \frac{\partial Q}{\partial K} - r = 0$ must hold in optimum.

Why? If the firm uses one unit of capital less and $\frac{\partial Q}{\partial K} / \frac{\partial Q}{\partial L}$ units of labour more, output remains (approximately) the same. The firm would save $r$ on capital and spend $\frac{\partial Q}{\partial K} / \frac{\partial Q}{\partial L} \times w$ on more labour while still making the same revenue. Profit would increase unless $r < \frac{\partial Q}{\partial K} / \frac{\partial Q}{\partial L} \times w$. As symmetric argument interchanging the role of capital and labour shows that $r \geq \frac{\partial Q}{\partial K} / \frac{\partial Q}{\partial L} \times w$ must hold if the firm optimizes profits.
Example

$$Q(K, L) = K^{\frac{1}{6}}L^{\frac{1}{2}}$$

$$\frac{\partial Q}{\partial K} = \frac{1}{6}K^{-\frac{5}{6}}L^{\frac{1}{2}}$$

$$\frac{\partial Q}{\partial L} = \frac{1}{2}K^{\frac{1}{6}}L^{-\frac{1}{2}}$$

FOC:

$$\frac{1}{6}K^{-\frac{5}{6}}L^{\frac{1}{2}} = \frac{r}{P}$$

$$\frac{\partial Q}{\partial K} / \frac{\partial Q}{\partial L} = \frac{1}{3}K^{-\frac{5}{6}}L^{\frac{1}{2}}(-\frac{1}{2}) = \frac{r}{w}$$

$$\frac{\partial Q}{\partial K} / \frac{\partial Q}{\partial L} = \frac{1}{3}K^{\frac{1}{6}}L^{-\frac{1}{2}} = \frac{r}{w}$$

Substituting $L = K$ into (*) we get

$$2K^{-\frac{5}{6}}K^{\frac{1}{6}} = 1$$

$$Q^* = K^{\frac{1}{6}}L^{\frac{1}{2}} = 2^\frac{1}{2} = 8$$

$$\Pi^* = 12Q^* - 1K^* - 3L^* = 48 - 8 - 24 = 16$$
Example 3: Price Discrimination

A monopolist with total cost function \( TC(Q) = Q^2 \) sells his product in two different countries. When he sells \( Q_A \) units of the good in country A he will obtain the price

\[ P_A = 22 - 3Q_A \]

for each unit. When he sells \( Q_B \) units of the good in country B he obtains the price

\[ P_B = 34 - 4Q_B. \]

How much should the monopolist sell in the two countries in order to maximize profits?

**Solution**

Total revenue in country A:

\[ TR_A = P_A Q_A = (22 - 3Q_A) Q_A \]

Total revenue in country B:

\[ TR_B = P_B Q_B = (34 - 4Q_B) Q_B \]

Total production costs are:

\[ TC = (Q_A + Q_B)^2 \]

Profit:

\[ \Pi(Q_A, Q_B) = (22 - 3Q_A) Q_A + (34 - 4Q_B) Q_B - (Q_A + Q_B)^2 \]

**FOC:**

\[
\frac{\partial \Pi}{\partial Q_A} = -3Q_A + (22 - 3Q_A) - 2(Q_A + Q_B) = 22 - 8Q_A - 2Q_B = 0
\]

\[
\frac{\partial \Pi}{\partial Q_B} = -4Q_B + (34 - 4Q_B) - 2(Q_A + Q_B) = 34 - 2Q_A - 10Q_B = 0
\]

or

\[
8Q_A + 2Q_B = 22 \quad \text{(6)}
\]

\[
2Q_A + 10Q_B = 34.
\]

*linear simultaneous system*

**Second order conditions: A peak**

Critical point of function \( z = f(x, y) = -x^2 - y^2 \): \((0, 0)\)
Second order conditions: A trough

critical point of function \( z = f(x, y) = +x^2 + y^2: \ (0, 0) \)

Second order conditions: A saddle point

critical point of function \( z = f(x, y) = +x^2 - y^2: \ (0, 0) \)

Second order conditions: A monkey saddle

critical point of function \( z = f(x, y) = yx^2 - y^3: \ (0, 0) \)

The Hessian matrix

\[
\begin{bmatrix}
\frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y\partial x} \\
\frac{\partial^2 z}{\partial x\partial y} & \frac{\partial^2 z}{\partial y^2}
\end{bmatrix}
\]
Theorem

Suppose that the function $z = f(x, y)$ has a critical point at $(x^*, y^*)$. If the determinant of the Hessian $\det H =$

$$
\begin{vmatrix}
\frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\
\frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2}
\end{vmatrix}
$$

is negative at $(x^*, y^*)$ then $(x^*, y^*)$ is a saddle point. If this determinant is positive at $(x^*, y^*)$ then $(x^*, y^*)$ is a peak or a trough. In this case the signs of $\frac{\partial^2 z}{\partial x^2}(x^*, y^*)$ and $\frac{\partial^2 z}{\partial y^2}(x^*, y^*)$ are the same. If both signs are positive, then $(x^*, y^*)$ is a peak. If both signs are negative, then $(x^*, y^*)$ is a trough.

- Nothing can be said if the determinant is zero.

Example

$z = x^2 - y^2$. The partial derivatives are $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = -2y$.

Clearly, $(0, 0)$ is the only critical point. The determinant of the Hessian is

$$
\det H = \begin{vmatrix}
\frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\
\frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2}
\end{vmatrix} = \begin{vmatrix}
2 & 0 \\
0 & -2
\end{vmatrix} = (2)(-2) - 0 \times 0 = -4 < 0
$$

Hence the function has a saddle point at $(0, 0)$. 

Example

\[ z = -x^2 - y^2. \] The partial derivatives are \( \frac{\partial z}{\partial x} = -2x \) and \( \frac{\partial z}{\partial y} = -2y. \) Clearly, (0, 0) is the only critical point. The determinant of the Hessian is

\[
\det H = \begin{vmatrix}
\frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\
\frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2}
\end{vmatrix} = \begin{vmatrix}
-2 & 0 \\
0 & -2
\end{vmatrix} = 4 > 0
\]

and \( \frac{\partial^2 z}{\partial x^2} < 0. \) Hence the function has a peak at (0, 0).

### Example

\[ z = -x^2 + \frac{5}{2}xy - y^2 \]

\[
\frac{\partial z}{\partial x} = -2x + \frac{5}{2}y \\
\frac{\partial z}{\partial y} = \frac{5}{2}x - 2y
\]

(0, 0) critical point.

Determinant of the Hessian is

\[
\det H = \begin{vmatrix}
\frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\
\frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2}
\end{vmatrix} = \begin{vmatrix}
-2 & \frac{5}{2} \\
\frac{5}{2} & -2
\end{vmatrix} = (-2)(-2) - \left(\frac{5}{2}\right)^2
\]

\[ = -\frac{9}{4} < 0 \]

Hence (0, 0) saddle point although both \( \frac{\partial^2 z}{\partial x^2} \) and \( \frac{\partial^2 z}{\partial y^2} \) negative.