| BEEM103 – Optimization Techiniques for Economists | Dieter Balkenborg<br>Departments of Economics |
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| Homework Week 6 | University of Exeter |

**Exercise 1** Find the solutions of the form $e^{\alpha x}$ to the homogenous equation

$$\ddot{x} - \dot{x} - 6x = 0$$

**Solution 1** The characteristic equation is

$$\alpha^2 - \alpha - 6 = (\alpha + 2)(\alpha - 3) = 0$$

The solutions are hence of the form

$$x(t) = Ae^{3t} + Be^{-2t}$$

**Exercise 2** Find a solution to the inhomogeneous equation

$$\ddot{x} - \dot{x} - 6x = 3$$

**Solution 2** $x(t) = -1/2$

**Exercise 3** Describe all solutions to this equation.

**Solution 3**

$$x(t) = Ae^{3t} + Be^{-2t} - 1/2$$

**Exercise 4** Find a solution with $x(0) = 0$, $x(1) = 1$

**Solution 4**

$$\begin{aligned} 0 &= x(0) = -1/2 + A + B \iff B = 1/2 - A \\ 1 &= x(1) = -1/2 + Ae^3 + (1/2 - A)e^{-2} = 1 \\ A &= \frac{1}{2}\frac{3 - e^{-2}}{e^3 - e^{-2}},\ B = -\frac{1}{2} - \frac{1}{2}\frac{3 - e^{-2}}{e^3 - e^{-2}} \end{aligned}$$

The solution is

$$x(t) = -\frac{1}{2} + \frac{1}{2}\frac{3 - e^{-2}}{e^3 - e^{-2}}e^{3t} - \left(\frac{1}{2} - + + \frac{1}{2}\frac{3 - e^{-2}}{e^3 - e^{-2}}\right)e^{-2t}$$

**Exercise 5** Minimize

$$\int_0^1 \left(t\dot{x} + \dot{x}^2\right) dt$$

subject to the boundary conditions $x(0) = 1$, $x(1) = 0$.

**Solution 5** Let $F(t, x, \dot{x}) = t\dot{x} + \dot{x}^2$

$$\begin{aligned} \frac{\partial F}{\partial x} &= 0,\ \frac{\partial F}{\partial \dot{x}} = t + 2\dot{x} \\ \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) &= 1 + 2\ddot{x} \end{aligned}$$

Euler equation:

$$\begin{aligned} 1 + 2\ddot{x} &= 0 \\ \ddot{x} &= -1/2 \\ \dot{x} &= -\frac{t}{2} + C \\ x &= -\frac{t^2}{4} + Ct + B \end{aligned}$$

$$
\begin{aligned}
x(0) &= 1 \Longleftrightarrow B = 1 \\
x(1) &= 0 \Longleftrightarrow -\frac{1}{4} + C + B = 0 \Longleftrightarrow C = \frac{1}{4} - 1 = -\frac{3}{4}
\end{aligned}
$$

The Euler equation and the boundary conditions yield the unique solution

$$x(t) = \frac{1}{4}t^2 - \frac{3}{4}t + 1$$

**Exercise 6** Minimize

$$\int_0^1 \left(x^2 + tx + tx\dot{x} + t^2\dot{x}^2\right) dt$$

subject to the boundary conditions $x(0) = 0$, $x(1) = 1$.

**Solution 6** Let $F = x^2 + tx + tx\dot{x} + t^2\dot{x}^2$. Then

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= 2x + t + t\dot{x},\ \frac{\partial F}{\partial \dot{x}} = tx + 2t^2\dot{x} \\
\frac{d}{dt}\frac{\partial F}{\partial \dot{x}} &= x + t\dot{x} + 4t\dot{x} + 2t^2\ddot{x}
\end{aligned}
$$

The Euler equation is hence

$$
\begin{aligned}
2x + t + t\dot{x} &= x + t\dot{x} + 4t\dot{x} + 2t^2\ddot{x} \\
x + t &= 4t\dot{x} + 2t^2\ddot{x} \\
2t^2\frac{d^2x}{dt^2} + 4t\frac{dx}{dt} - x - t &= 0
\end{aligned}
$$

We must first find a special solution of this inhomogenous linear equation with non-constant coefficients. Here trying a linear function $x(t) = \alpha t$ does the trick:

$$
\begin{aligned}
\frac{dx}{dt} &= \alpha,\ \frac{d^2x}{dt^2} = 0 \\
2t^2\frac{d^2x}{dt^2} + 4t\frac{dx}{dt} - x - t &= 4\alpha t - \alpha t - t = (-3\alpha - 1)t = 0 \Longleftrightarrow \alpha = -\frac{1}{3}
\end{aligned}
$$

To find a solution to the homogeneous equation

$$2t^2\frac{d^2x}{dt^2} + 4t\frac{dx}{dt} - x = 0$$

we set $x(t) = t^r$ and obtain

$$
\begin{aligned}
& 2t^2 r(r-1)t^{r-2} + 4trt^{t-1} - t^r = 0 \\
\Longleftrightarrow\ & 2r(r-1) + 4r - 1 = 0 \\
\Longleftrightarrow\ & r = \pm\frac{\sqrt{3}}{2} - \frac{1}{2}
\end{aligned}
$$

, Solution is: $\frac{1}{2}\sqrt{3} - \frac{1}{2}, -\frac{1}{2}\sqrt{3} - \frac{1}{2}$We obtain the general solution to the Euler equation

$$At^{\frac{\sqrt{3}}{2}-\frac{1}{2}} + Bt^{-\frac{\sqrt{3}}{2}-\frac{1}{2}} - \frac{1}{3}t$$

Finally, the boundary conditions must be satisfied. Unluckily I have taken here again $x(0) = 0$, which is a problem because $t^r$ is generally not defined at $t = 0$. We would have to take $B = 0$ to have the function defined and then $A = 0$ to satisfy $x(0) = 0$. But then the resulting function $-t/3$ would not satisfy $x(1) = 1$ and so no solution exists.

Notice that the boundary conditions $x(1) = 0$, $x(2) = 0$ would yield a solution: One would get the equations

$$\begin{aligned} A + B - \frac{1}{3} &= 0 \\ A \times 2^{-\frac{3}{2}+\frac{1}{2}\sqrt{13}} + B \times 2^{-\frac{3}{2}-\frac{1}{2}\sqrt{13}} - \frac{2}{3} &= 0 \end{aligned}$$

with the solution

$$\begin{aligned} A &= -\frac{1}{3}\frac{-2+2^{-\frac{3}{2}-\frac{1}{2}\sqrt{13}}}{2^{-\frac{3}{2}+\frac{1}{2}\sqrt{13}}-2^{-\frac{3}{2}-\frac{1}{2}\sqrt{13}}} \\ B &= \frac{1}{3}\frac{2^{-\frac{3}{2}+\frac{1}{2}\sqrt{13}}-2}{2^{-\frac{3}{2}+\frac{1}{2}\sqrt{13}}-2^{-\frac{3}{2}-\frac{1}{2}\sqrt{13}}} \end{aligned}$$

and so we would obtain the function

$$\begin{aligned} &x(t) = At^{-\frac{3}{2}+\frac{1}{2}\sqrt{13}} + Bt^{-\frac{3}{2}-\frac{1}{2}\sqrt{13}} - \frac{1}{3}t = \\ &-\frac{1}{3}\frac{-2+2^{-\frac{3}{2}-\frac{1}{2}\sqrt{13}}}{2^{-\frac{3}{2}+\frac{1}{2}\sqrt{13}}-2^{-\frac{3}{2}-\frac{1}{2}\sqrt{13}}}t^{-\frac{3}{2}+\frac{1}{2}\sqrt{13}} + \frac{1}{3}\frac{2^{-\frac{3}{2}+\frac{1}{2}\sqrt{13}}-2}{2^{-\frac{3}{2}+\frac{1}{2}\sqrt{13}}-2^{-\frac{3}{2}-\frac{1}{2}\sqrt{13}}}t^{-\frac{3}{2}-\frac{1}{2}\sqrt{13}} - \frac{1}{3}t \end{aligned}$$