**Exercise 1** Find the solutions of the form $e^{\alpha x}$ to the homogenous equation

$$\ddot{x} - \dot{x} - 6x = 0$$

**Solution 1** The characteristic equation is

$$\alpha^2 - \alpha - 6 = (\alpha + 2)(\alpha - 3) = 0$$

The solutions are hence of the form

$$x(t) = Ae^{3t} + Be^{-2t}$$

**Exercise 2** Find a solution to the inhomogeneous equation

$$\ddot{x} - \dot{x} - 6x = 3$$

**Solution 2** $x(t) = -1/2$

**Exercise 3** Describe all solutions to this equation.

**Solution 3**

$$x(t) = Ae^{3t} + Be^{-2t} - 1/2$$

**Exercise 4** Find a solution with $x(0) = 0$, $x(1) = 1$

**Solution 4**

$$0 = x(0) = -\frac{1}{2} + A + B \iff B = \frac{1}{2} - A$$

$$1 = x(1) = -\frac{1}{2} + Ae^3 + (\frac{1}{2} - A)e^{-2} = 1$$

$$A = \frac{1}{2} \frac{3 - e^{-2}}{e^3 - e^{-2}}, \quad B = -\frac{1}{2} - \frac{1}{2} \frac{3 - e^{-2}}{e^3 - e^{-2}}$$

The solution is

$$x(t) = \frac{1}{2} + \frac{1}{2} \frac{3 - e^{-2}}{e^3 - e^{-2}} e^{3t} - \left(\frac{1}{2} - + \frac{1}{2} \frac{3 - e^{-2}}{e^3 - e^{-2}}\right) e^{-2t}$$

**Exercise 5** Minimize

$$\int_0^1 (t\dot{x} + \dot{x}^2) \, dt$$

subject to the boundary conditions $x(0) = 1$, $x(1) = 0$.

**Solution 5** Let $F(t, x, \dot{x}) = t\dot{x} + \dot{x}^2$

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial \dot{x}} = t + 2\dot{x}$$

$$\frac{d}{dt} \left( \frac{\partial F}{\partial \ddot{x}} \right) = 1 + 2\ddot{x}$$

Euler equation:

$$1 + 2\ddot{x} = 0$$

$$\ddot{x} = -1/2$$

$$\dot{x} = -\frac{t}{2} + C$$

$$x = -\frac{t^2}{4} + Ct + B$$
\[ x(0) = 1 \iff B = 1 \]
\[ x(1) = 0 \iff -\frac{1}{4} + C + B = 0 \iff C = \frac{1}{4} - 1 = -\frac{3}{4} \]

The Euler equation and the boundary conditions yield the unique solution

\[ x(t) = \frac{1}{4} t^2 - \frac{3}{4} t + 1 \]

**Exercise 6** Minimize

\[ \int_0^1 (x^2 + tx + tx\dot{x} + t^2 \ddot{x}^2) \, dt \]

subject to the boundary conditions \( x(0) = 0, x(1) = 1 \).

**Solution 6** Let \( F = x^2 + tx + tx\dot{x} + t^2 \ddot{x}^2 \). Then

\[ \frac{\partial F}{\partial x} = 2x + t + t\dot{x}, \quad \frac{\partial F}{\partial \dot{x}} = tx + 2t^2 \ddot{x} \]

\[ \frac{d}{dt} \frac{\partial F}{\partial \ddot{x}} = x + t\dot{x} + 4t\dot{x} + 2t^2 \ddot{x} \]

The Euler equation is hence

\[ 2x + t + t\dot{x} = x + t\dot{x} + 4t\dot{x} + 2t^2 \ddot{x} \]
\[ x + t = 4t\dot{x} + 2t^2 \ddot{x} \]
\[ 2t^2 \frac{d^2 x}{dt^2} + 4t \frac{dx}{dt} - x - t = 0 \]

We must first find a special solution of this inhomogenous linear equation with non-constant coefficients. Here trying a linear function \( x(t) = \alpha t \) does the trick:

\[ \frac{dx}{dt} = \alpha, \quad \frac{d^2 x}{dt^2} = 0 \]
\[ 2t^2 \frac{d^2 x}{dt^2} + 4t \frac{dx}{dt} - x - t = 4\alpha t - \alpha t - t = (-3\alpha - 1)t = 0 \iff \alpha = -\frac{1}{3} \]

To find a solution to the homogeneous equation

\[ 2t^2 \frac{d^2 x}{dt^2} + 4t \frac{dx}{dt} - x = 0 \]
we set \( x(t) = t^r \) and obtain

\[ 2t^2 r (r - 1) t^{r-2} + 4t r t^{r-1} - t^r = 0 \]
\[ \iff 2r (r - 1) + 4r - 1 = 0 \]
\[ \iff r = \frac{\pm \sqrt{3}}{2} - \frac{1}{2} \]

, Solution is: \( \frac{1}{2} \sqrt{3} - \frac{1}{2}, -\frac{1}{2} \sqrt{3} - \frac{1}{2} \) We obtain the general solution to the Euler equation

\[ Ax^{\frac{\sqrt{3}}{2} - \frac{1}{2}} + Bt^{-\frac{\sqrt{3}}{2} - \frac{1}{2}} - \frac{1}{3} t \]

Finally, the boundary conditions must be satisfied. Unluckily I have taken here again \( x(0) = 0 \), which is a problem because \( t^r \) is generally not defined at \( t = 0 \). We would have to take \( B = 0 \) to have the function defined and then \( A = 0 \) to satisfy \( x(0) = 0 \). But then the resulting function \(-t/3\) would not satisfy \( x(1) = 1\) and so no solution exists.
Notice that the boundary conditions \( x(1) = 0, x(2) = 0 \) would yield a solution: One would get the equations

\[
A + B - \frac{1}{3} = 0 \\
A \times 2^{-\frac{2}{5} + \frac{1}{2} \sqrt{13}} + B \times 2^{-\frac{2}{5} - \frac{1}{2} \sqrt{13}} - \frac{2}{3} = 0
\]

with the solution

\[
A = -\frac{1}{3} \frac{-2 + 2^{-\frac{2}{5} - \frac{1}{2} \sqrt{13}}}{2^{-\frac{2}{5} + \frac{1}{2} \sqrt{13}} - 2^{-\frac{2}{5} - \frac{1}{2} \sqrt{13}}} \\
B = \frac{1}{3} \frac{2^{-\frac{2}{5} + \frac{1}{2} \sqrt{13}} - 2}{2^{-\frac{2}{5} + \frac{1}{2} \sqrt{13}} - 2^{-\frac{2}{5} - \frac{1}{2} \sqrt{13}}}
\]

and so we would obtain the function

\[
x(t) = At^{\frac{2}{5} + \frac{1}{2} \sqrt{13}} + Bt^{\frac{2}{5} - \frac{1}{2} \sqrt{13}} - \frac{1}{3} t = \\
-\frac{1}{3} \frac{-2 + 2^{-\frac{2}{5} - \frac{1}{2} \sqrt{13}}}{2^{-\frac{2}{5} + \frac{1}{2} \sqrt{13}} - 2^{-\frac{2}{5} - \frac{1}{2} \sqrt{13}}} t^{\frac{2}{5} + \frac{1}{2} \sqrt{13}} + \frac{1}{3} \frac{2^{-\frac{2}{5} + \frac{1}{2} \sqrt{13}} - 2}{2^{-\frac{2}{5} + \frac{1}{2} \sqrt{13}} - 2^{-\frac{2}{5} - \frac{1}{2} \sqrt{13}}} t^{\frac{2}{5} - \frac{1}{2} \sqrt{13}} - \frac{1}{3} t
\]