BEEM103 – Optimization Techniques for Economists	Dieter Balkenborg Departments of Economics
Homework Week 3 – SOLUTIONS	University of Exeter

Exercise 1 Show that the function

$$Q = K^{\alpha} L^{\beta}$$

with $\alpha, \beta > 0$ and $\alpha + \beta < 1$ is concave.

Solution 1 It is assmed here that K and L are positive.

$$\frac{\partial Q}{\partial K} = \alpha K^{\alpha - 1} L^{\beta},\ \frac{\partial Q}{\partial L} = \beta K^{\alpha} L^{\beta - 1}$$

$$H = \begin{bmatrix} \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial L \partial K} \\ \frac{\partial^2 Q}{\partial K \partial L} & \frac{\partial^2 Q}{\partial L^2} \end{bmatrix} = \begin{bmatrix} \alpha (\alpha - 1) K^{\alpha - 2} L^{\beta} & \alpha\beta K^{\alpha - 1} L^{\alpha - 1} \\ \alpha\beta K^{\alpha - 1} L^{\alpha - 1} & \beta (\beta - 1) K^{\alpha} L^{\beta - 2} \end{bmatrix}$$

Since $0 < \alpha < 1$ we have $\frac{\partial^2 Q}{\partial K^2} = \alpha (\alpha - 1) K^{\alpha - 2} L^{\beta} < 0$. Moreover

$$\begin{aligned} \det H &= \left[\alpha (\alpha - 1) \beta (\beta - 1) - \alpha^2 \beta^2\right] K^{2\alpha - 2} L^{2\beta - 2} \\ &= \left[\alpha\beta (\alpha\beta - \alpha - \beta + 1) - \alpha^2 \beta^2\right] K^{2\alpha - 2} L^{2\beta - 2} \\ &= \left[\alpha\beta (1 - \alpha - \beta)\right] K^{2\alpha - 2} L^{2\beta - 2} > 0 \end{aligned}$$

Hence the function is concave.

Exercise 2 Use the least-square criterion to find the equation of the line that is closest to the points $(1, 1)$, $(2, 3)$ and $(4, 3)$. That is, find the line $y = f(x) = ax + b$ which minimizes

$$F(a, b) = \sum_{i=1}^{n} (y_i - f(x_i))^2$$

where the summation is over the above three data points (x_i, y_i).

Solution 2 The first order-conditions are with

$$n\bar{x} = \sum_{i=1}^{n} x_i,\ n\bar{y} = \sum_{i=1}^{n} y_i,\ nv = \sum_{i=1}^{n} x_i^2,\ nc = \sum_{i=1}^{n} x_i y_i$$

$$\begin{aligned} \frac{\partial F}{\partial b} &= 2 \sum_{i=1}^{n} (y_i - a x_i - b) = 0 \iff a\bar{x} + b = \bar{y} \\ \frac{\partial F}{\partial a} &= 2 \sum_{i=1}^{n} (y_i - a x_i - b)(-x_i) = 0 \iff av + b\bar{x} = c \end{aligned}$$

The solution is given by

$$\begin{aligned} a\bar{x}^2 + b\bar{x} &= \bar{x}\bar{y} \\ av + b\bar{x} &= c \\ a\left(v - \bar{x}^2\right) &= c - \bar{x}\bar{y} \\ a &= \\ b &= \bar{y} - a\bar{x} = \bar{y} - \frac{c - \bar{x}\bar{y}}{(v - \bar{x}^2)} \bar{x} \end{aligned}$$

Now put in the numbers to get $a = \frac{4}{7}$ and $b = 1$ in our example.

Exercise 3 The highway department is planning to build a picnic area for motorists along a major highway. It is to be rectangular with an area of $5,000$ square yards and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing that will be needed to complete the job?

a) Identify this problem as a constraint optimization problem. What objective function $f(x,y)$ is to be maximized / minimized subject to what constraint $g(x,y) \geq 0$?

b) Write down the Lagrangian $\mathcal{L}(x,y)$ for this problem.

c) Find the solution to the three equations a) $\frac{\partial \mathcal{L}}{\partial x} = 0$ b) $\frac{\partial \mathcal{L}}{\partial y} = 0$ c) $g(x,y) \geq 0$.

Solution 3 a) Let x denote the length of the side of the rectangle adjacent to the highway and let y denote the length of the other side. Then we have to minimize the amount of fencing $x + 2y$ subject of the constraint that the area xy is at least 5000 ($xy \geq 5000$). Equivalently, we have to maximize $-x - 2y$ subject to the constraint $xy - 5000 \geq 0$.

b) The Lagrangian is

$$\mathcal{L}(x,y) = -x - 2y + \lambda(xy - 5000)$$

c)

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= -1 + \lambda y = 0 \qquad \text{or} \qquad 1 = \lambda y \\
\frac{\partial \mathcal{L}}{\partial y} &= -2 + \lambda x = 0 \qquad \text{or} \qquad 2 = \lambda x \\
5000 &= xy
\end{aligned}$$

Division of the two equations on the right yields $\frac{1}{2} = \frac{\lambda y}{\lambda x} = \frac{y}{x}$ or $x = 2y$. Hence $5000 = xy = 2y^2$, $y^2 = 2500$, $y = 50$ (since the solution -50 does not make sense). We obtain $x^* = 100$, $y^* = 50$ and $\lambda = 1/50 \geq 0$.

Exercise 4 A consumer has the following utility when he consumes x units of apples and y units of oranges:

$$u(x,y) = -x^2 + 4x - y^2 + 16y$$

Suppose the consumer has a budget of £3.20 to be spend on oranges and apples. Each apple and each orange costs £0.40. Use the method of Lagrange to find the optimal consumption bundle:
a) Write down the budget constraint and the Lagrangian. Assume that only the budget constraint is binding.

Solution 4 a) With the given prices and budget the budget constraint is

$$0.4x + 0.4y = 3.2.$$

The Lagrangian for the constrained utility maximization problem is hence

$$\begin{aligned}
\mathcal{L}(x,y) &= u(x,y) + \lambda[3.2 - 0.4x - 0.4y] + \lambda_2 x + \lambda_3 y \\
&= -x^2 + 4x - y^2 + 16y + \lambda[3.2 - 0.4x - 0.4y]
\end{aligned}$$

since $\lambda_2 = \lambda_3 = 0$ by the complementarity conditionsb) In the optimum the two partial derivatives of the Lagrangian must be zero:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= -2x + 4 - \lambda[0.4] = 0 \qquad \text{or} \qquad -2x + 4 = 0.4\lambda \\
\frac{\partial \mathcal{L}}{\partial x} &= -2y + 16 - \lambda[0.4] = 0 \qquad \text{or} \qquad -2y + 16 = 0.4\lambda
\end{aligned}$$

Division of the two equations on the right yields

$$\begin{aligned}
\frac{-2x+4}{-2y+16} &= \frac{0.4\lambda}{0.4\lambda} = 1 \qquad \text{or} \qquad -2x + 4 = -2y + 16 \qquad \text{or} \\
-2x - 12 &= -2y \qquad \text{or} \qquad y = x + 6
\end{aligned}$$

Thus $y = x + 6$ must hold at the constrained utility maximum.
c) In addition, the constraint

$$\begin{aligned} 0.4x + 0.4y &= 3.2 \qquad \text{or} \qquad x + y = \frac{3.2}{0.4} = 8 \qquad \text{or} \\ y &= 8 - x \end{aligned}$$

must hold at the optimum. Overall $y = x + 6$ and $y = 8 - x$ must hold, so

$$x + 6 = 8 - x \qquad \text{or} \qquad 2x = 2 \qquad \text{or} \qquad x = 1$$

Moreover, $x = 1$ implies $y = 8 - x = 7$. *Thus the Lagrangian approach suggests that it is optimal for the consumer to buy one apple and seven oranges.* (Notice that we obtain $0.4\lambda = 16 - 2y = 16 - 14 = 2 > 0$ for the Lagrange multiplier.)