BEEM103 – Optimization Techniques for Economists	Dieter Balkenborg Departments of Economics
Homework Week 2 – SOLUTIONS	University of Exeter

Exercise 1 Calculate the first and second order partial derivatives of

$$
\begin{aligned}
\text{a) } z &= (2-x-y)\,x + \left(5+2x-3y^2\right)y - 3x + 2y^2 \\
\text{b) } z &= \left(x^2+y^3\right)^5
\end{aligned}
$$

Solution 1 a)

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= (2-x-y) - x + 2y - 3 = -1 - 2x + y \\
\frac{\partial z}{\partial y} &= -x + 5 + 2x - 3y^2 - 6y^2 + 4y = x + 5 - 9y^2 + 4y
\end{aligned}
$$

$$
\begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -18y + 4 \end{bmatrix}
$$

b)

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= 5\left(x^2+y^3\right)^4 \times 2x = 10x\left(x^2+y^3\right)^4 \\
\frac{\partial z}{\partial y} &= 5\left(x^2+y^3\right)^4 \times 3y^2 = 15y^2\left(x^2+y^3\right)^4
\end{aligned}
$$

$$
\begin{aligned}
&\begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} \\
= &\begin{bmatrix} 10\left(x^2+y^3\right)^4 + 40x\left(x^2+y^3\right)^3 \times 2x & 10x \times 4\left(x^2+y^3\right)^3 \times 3y^2 \\ 15y^2 \times 4\left(x^2+y^3\right)^3 \times 2x & 15\left(x^2+y^3\right)^4 + 15y^2\left(x^2+y^3\right)^3 \times 3y \end{bmatrix} \\
= &\left(x^2+y^3\right)^3 \begin{bmatrix} 10\left(x^2+y^3\right) + 80x^2 & 120xy^2 \\ 30xy^2 & 15\left(x^2+y^3\right) + 45y^3 \end{bmatrix} \\
= &\left(x^2+y^3\right)^3 \begin{bmatrix} 90x^2 + 10y^3 & 30xy^2 \\ 120xy^2 & 15x^2 + 60y^3 \end{bmatrix}
\end{aligned}
$$

Exercise 2 Find the critical point of the functions

$$
\begin{aligned}
\text{a) } z &= f(x,y) = xy - 2x + 3y - 6 \\
\text{b) } z &= g(x,y) = 2x^2 + 2xy - 6x + 5y^2 - 6y + 5 \\
\text{c) } z &= h(x,y) = -2x^2 - 2xy + 6x - 5y^2 - 5
\end{aligned}
$$

Determine whether they are troughs, peaks or saddle points.

Solution 2 a)

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= y - 2 = 0 \Leftrightarrow y = 2 \\
\frac{\partial z}{\partial y} &= x + 3 = 0 \Leftrightarrow x = -3
\end{aligned}
$$

The critical point is $(x,y) = (3,2)$.

$$
\det H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 < 0
$$

The critical point is hence a saddle point.

b)

$$\begin{aligned} \frac{\partial z}{\partial x} &= 4x + 2y - 6 = 0 \\ \frac{\partial z}{\partial y} &= 2x + 10y - 6 = 0 | \times 2 \end{aligned}$$

$$\begin{aligned} 4x + 2y &= 6 | - \\ 4x + 20y &= 12 | + \\ -18y &= -6 \end{aligned}$$

Hence $y = \frac{1}{3}$, $4x = 6 - \frac{2}{3} = \frac{16}{3}$, $x = \frac{4}{3}$. The critical point is $(x, y) = \left(\frac{4}{3}, \frac{1}{3}\right)$. Since

$$\det H = \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix} = 40 - 4 = 36 > 0$$

the critical point is *not* a saddle point. Since $\frac{\partial^2 z}{\partial x^2} = 4 > 0$ it is a trough. (In fact, the function is strictly convex and the critical point a global minimum.)

c)

$$\begin{aligned} \frac{\partial z}{\partial x} &= -4x - 2y + 6 = 0 \\ \frac{\partial z}{\partial y} &= -2x - 10y = 0 | \times (-2) \end{aligned}$$

$$\begin{aligned} -4x - 2y &= -6 | + \\ 4x + 20y &= 0 | + \\ 18y &= -6 \end{aligned}$$

Hence $y = -\frac{1}{3}$, $4x = 6 + \frac{2}{3} = \frac{20}{3}$, $x = \frac{5}{3}$. The critical point is again $(x, y) = \left(\frac{5}{3}, -\frac{1}{3}\right)$. Since

$$\det H = \begin{bmatrix} -4 & -2 \\ -2 & -10 \end{bmatrix} = 40 - 4 = 36 > 0$$

the critical point is *not* a saddle point. Since $\frac{\partial^2 z}{\partial x^2} = -4 < 0$ it is a peak. (In fact, the function is strictly concave and the critical point a global maximum.)

Exercise 3 A dairy produces whole milk and skim milk in quantities x and y gallons, respectively. Suppose that the price of whole milk is $p(x) = 20 - 5x$ and that the price of skim milk is $q(y) = 4 - 2y$ and assume that $C(x, y) = 2xy + 4$ is the total (!) joint-cost function of the commodities. What should x and y be to maximize profit, assuming that the first order conditions yield a maximum? Show that your solution is indeed the unique maximum.

Solution 3 The revenue from selling whole milk is

$$xp(x) = x(20 - 5x),$$

the revenue from selling skim milk is

$$yq(y) = y(4 - 2y)$$

and the total costs are $2xy + 4$. The profit is therefore

$$\Pi(x, y) = x(20 - 5x) + y(4 - 2y) - 2xy - 4$$

The conditions for a critical point are

$$\begin{aligned}\frac{\partial \Pi}{\partial x} &= 20 - 5x - 5x - 2y = 20 - 10x - 2y = 0 \\ \frac{\partial \Pi}{\partial y} &= 4 - 2y - 2y - 2x = 4 - 2x - 4y = 0\end{aligned}$$

Dividing the second equations by 2 we obtain the simultaneous system of equations

$$\begin{aligned}10x + 2y &= 20 \\ x + 2y &= 2\end{aligned}$$

Subtraction yields $9x = 18$ or $x = 2$ and so we obtain from the second equation $y = 1 - \frac{1}{2}x = 1 - 1 = 0$. Thus the profit function has a critical point at $x = 2$ and at $y = 0$.