BEEM103 – Optimization Techniques for Economists	Dieter Balkenborg Departments of Economics
Class Exercises Week 6 - Solutions	University of Exeter

Exercise 1 Find the solutions of the form $e^{\alpha t}$ to the homogenous equation

$$\ddot{x} - 2\dot{x} + x = 0$$

Solution 1 For such a solution $\dot{x} = \alpha e^{\alpha t}$, $\ddot{x} = \alpha^2 e^{\alpha t}$ and hence

$$\ddot{x} - 2\dot{x} + x = \left[\alpha^2 - 2\alpha + 1\right] e^{\alpha t} = (\alpha - 1)^2 e^{\alpha t}$$

A solution is given by $\alpha = 1$, i.e. $x(t) = e^t$.

Exercise 2 Find a solution to the inhomogeneous equation

$$\ddot{x} - 2\dot{x} + x = 3$$

Solution 2 A solution is given by $x(t) = 3$.

Exercise 3 Describe all solutions to this equation.

Solution 3 Because $\alpha^2 - 2\alpha + 1$ has the double root $\alpha = 1$ it follows from Theorem 6.3.1 in the textbook by Sydsæter, Hammond, Seierstad and Strøm that the general solution to the homogeneous equation $\ddot{x} - 2\dot{x} + x = 0$ is $(A + Bt) e^t$. The same book discusses two pages later that the general solution to the inhomogeneous solution is

$$(A + Bt) e^t + 3$$

Exercise 4 Find a solution with $x(0) = 0$, $x(1) = 1$

Solution 4

$$\begin{aligned} x(0) &= A + 3 = 0 \Longleftrightarrow A = -3 \\ x(1) &= (-3 + B) e + 3 = 2 \Longleftrightarrow -3 + B = -e^{-1} \Longleftrightarrow B = -e^{-1} + 3 \end{aligned}$$

Exercise 5 Find the solutions to the homogenous equation

$$\ddot{x} + x = 0$$

Solution 5 The characteristic equation $\alpha^2 + 1$ has no real solution. By Theorem 6.3.1 in the textbook by Sydsæter, Hammond, Seierstad and Strøm the general solution is

$$x = A \cos t + B \sin t$$

Exercise 6 Minimize

$$\int_0^1 \sqrt{1 + \dot{x}^2} dt$$

subject to the boundary conditions $x(0) = 4$, $x(1) = 2$.

Solution 6 Let $F = \sqrt{1+\dot{x}^2}$. We have

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 0,\ \frac{\partial F}{\partial \dot{x}} = \frac{1}{2}\left(1+\dot{x}^2\right)^{-\frac{1}{2}} \times 2\dot{x} = \left(1+\dot{x}^2\right)^{-\frac{1}{2}} \times \dot{x} \\
\frac{\partial^2 F}{\partial \dot{x}\partial t} &= \frac{\partial^2 F}{\partial \dot{x}\partial x} = 0 \\
\frac{\partial^2 F}{\partial \dot{x}\partial \dot{x}} &= -\frac{1}{2}\left(1+\dot{x}^2\right)^{-\frac{3}{2}} \times 2\dot{x} \times \dot{x} + \left(1+\dot{x}^2\right)^{-\frac{1}{2}} \\
&= \frac{1}{\sqrt{1+\dot{x}^2}}\left(-\frac{\dot{x}^2}{1+\dot{x}^2}+1\right) = \frac{1}{\sqrt{1+\dot{x}^2}}\frac{1}{1+\dot{x}^2} > 0
\end{aligned}$$

The Euler equation is hence in this case

$$0 = \frac{1}{\sqrt{1+\dot{x}^2}}\frac{1}{1+\dot{x}^2}\ddot{x} \Longleftrightarrow \ddot{x} = 0$$

The general solution to $\ddot{x} = 0$ is $x = At + B$. To fit the boundary conditions we must have

$$\begin{aligned}
x(0) &= B = 4 \\
x(1) &= A + 4 = 2 \Leftrightarrow A = -2
\end{aligned}$$

Thus $x(t) = -2t + 4$

Exercise 7 Minimize

$$\int_0^1 \left(x^2 + 2tx\dot{x} + t^2\dot{x}^2\right) dt$$

subject to the boundary conditions $x(0) = 0$, $x(1) = 2$.

Solution 7

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2x + 2t\dot{x},\ \frac{\partial F}{\partial \dot{x}} = 2tx + 2t^2\dot{x} \\
\frac{\partial^2 F}{\partial \dot{x}\partial t} &= 2x + 4t\dot{x},\ \frac{\partial^2 F}{\partial \dot{x}\partial x} = 2t,\ \frac{\partial^2 F}{\partial \dot{x}\partial \dot{x}} = 2t^2
\end{aligned}$$

The Euler equation is

$$\begin{aligned}
2x + 2t\dot{x} &= 2x + 4t\dot{x} + 2t\dot{x} + 2t^2\ddot{x} \\
0 &= 4t\dot{x} + 2t^2\ddot{x} \\
&\Longleftrightarrow \ddot{x} = -\frac{2}{t}\dot{x} \text{ for } t \neq 0
\end{aligned}$$

Let $u = \dot{x}$. Then the last equation becomes

$$\frac{du}{dt} = -\frac{2}{t}u$$

This is a separable differential equation which we can solve as follows

$$\begin{aligned}
\frac{1}{u}du &= -\frac{2}{t}dt \\
\int \frac{1}{u}du &= -\int \frac{2}{t}dt \\
\ln|u| &= -2\ln|t| + C \\
\ln|u| &= \ln(|t|)^{-2} + C = \ln\left(\alpha t^{-2}\right) \\
|u| &= \alpha t^{-2} \text{ or } u = \pm\alpha t^{-2}
\end{aligned}$$

where $\alpha = \ln C$.

Next

$$\begin{aligned}
\frac{dx}{dt} &= u = \pm\alpha t^{-2} \\
x &= \pm\alpha \int t^{-2}dt = \mp\alpha t^{-1} + D
\end{aligned}$$

The only function possible which could in addition satisfy $x(0) = 0$ would be obtained for $\alpha = 0$ (otherwise the function would not be defined at zero) and $A = 0$, i.e., we would have $x(t) = 0$ for all t. However, $x(2) = 2$ would then not be possible. Hence we cannot find a solution and suspect that none exists.

Exercise 8 Find the Euler equation for the Ramsey problem

$$\int_0^T U\left(f(K(t)) - \dot{K}(t)\right)e^{-rt}dt$$

subject to the boundary conditions $x(0) = 4$, $x(1) = 2$. Use the relative rate of change consumption $\dot{C}/C$ where $C = f(K) - \dot{K}$.

Solution 8 For this and the next exercise see Sydsæter, Hammond, Seierstad and Strøm Section 8.4.

Exercise 9 Solve the equation in the case $f(K) = bK$, $U(C) = C^{1-v}/(1-v)$.