BEEM103 – Optimization Techniques for Economists	Dieter Balkenborg Departments of Economics
Class Exercises Week 2 – Solutions	University of Exeter

Exercise 1 Calculate the partial derivatives of

$$\begin{aligned} \text{a) } z &= 5y^5 + 4x^4y + 3x^2y^3 + 2xy^4 + 2x + 3y + 5 \\ \text{b) } z &= \frac{xy^2}{x^2y^3+1} \\ \text{c) } z &= \left(x^9y+1\right)\left(xy^8+1\right) \end{aligned}$$

Solution 1

$$\begin{aligned} \text{a) } \frac{\partial z}{\partial x} &= 16x^3y + 6xy^3 + 2y^4 + 2 \\ \frac{\partial z}{\partial y} &= 25y^4 + 4x^4 + 9x^2y^2 + 8xy^3 + 3 \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{\partial z}{\partial x} &= \frac{y^2\left(x^2y^3+1\right) - xy^2\left(2xy^3\right)}{\left(x^2y^3+1\right)^2} = \frac{y^2\left(1-x^2y^3\right)}{\left(x^2y^3+1\right)^2} \\ \frac{\partial z}{\partial y} &= \frac{2xy\left(x^2y^3+1\right) - xy^2\left(3x^2y^2\right)}{\left(x^2y^3+1\right)^2} = \frac{xy\left(2-x^2y^3\right)}{\left(x^2y^3+1\right)^2} \end{aligned}$$

$$\begin{aligned} \text{c) } \frac{\partial z}{\partial x} &= 9x^8y\left(xy^8+1\right) + \left(x^9y+1\right)y^8 \\ \frac{\partial z}{\partial y} &= x^9y\left(xy^8+1\right) + \left(x^9y+1\right) \times 8xy^7 \end{aligned}$$

Exercise 2 Find all second derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ of

$$z = 5x^2y + 3x^2y^2 + 5y^3$$

Solution 2

$$\begin{aligned} \frac{\partial z}{\partial x} &= 10xy + 6xy^2 \\ \frac{\partial z}{\partial y} &= 5x^2 + 6x^2y + 15y^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= 10y + 6y^2 \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 10x + 12xy \\ \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 10x + 12xy \\ \frac{\partial^2 z}{\partial y^2} &= 6x^2 + 30y \end{aligned}$$

Exercise 3 Find the equations $z = ax + c$ for the tangents to the graph of the function

$$z = f(x) = x^3 - 4x$$

for $x_0 = 1$ and for $x_0 = 3$.

Hint: Use the formula

$$dz = f'(x_0)\, dx$$

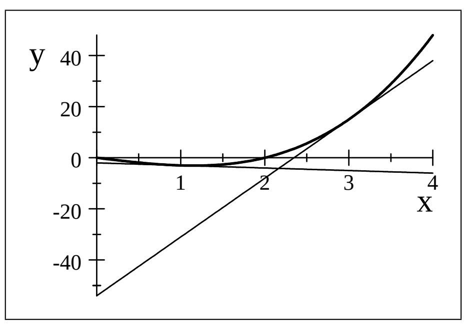
where $dx = x - x_0$, $dz = z - z_0$ and $z_0 = f(x_0)$.

Solution 3 $f(1) = -3$; $f(3) = 15$; $f'(x) = 3x^2 - 4$, $f'(1) = -1$, $f'(3) = 23$. The tangents at $x_0 = 1$ and $x_0 = 3$ are hence given by

$$\begin{aligned} dz &= (-1)\, dx \\ (z - (-3)) &= (-1)(x - 1) \\ z &= -x + 1 - 3 = -x - 2 \end{aligned}$$

$$\begin{aligned} dz &= (23)\, dx \\ (z - 15) &= 23(x - 3) \\ z &= 23x + 15 - 69 = 23x - 54 \end{aligned}$$

This fits with the following graph



Exercise 4 Find the equations $z = ax + by + c$ for the tangent planes to the graph of the function

$$z = f(x, y) = z(x, y)\, z = x^3 y - 4xy^2$$

for $(x_0, y_0) = (1, 2)$ and for $(x_0, y_0) = (2, 1)$.

Hint: Use the formula

$$dz = \frac{\partial z}{\partial x}_{|x=x_0, y=y_0} dx + \frac{\partial z}{\partial y}_{|x=x_0, y=y_0} dy$$

where $dx = x - x_0$, $dy = y - y_0$, $dz = z - z_0$ and $z_0 = f(x_0)$.

Solution 4 $z(1,2) = -14$; $z(2,1) = 0$

$$\begin{aligned}\frac{\partial z}{\partial x} &= 3x^2y - 4y^2;\ \frac{\partial z}{\partial x}_{|x=1,y=2} = -10;\ \frac{\partial z}{\partial x}_{|x=2,y=1} = 8\\ \frac{\partial z}{\partial y} &= b(x,y) = x^3 - 8xy;\ \frac{\partial z}{\partial x}_{|x=1,y=2} = -15;\ \frac{\partial z}{\partial x}_{|x=2,y=1} = -8\end{aligned}$$

The equation for the tangent plane at $(1,2)$ is hence given by

$$\begin{aligned}dz &= \frac{\partial z}{\partial x}_{|x=2,y=1} dx + \frac{\partial z}{\partial y}_{|x=2,y=1} dy = -10dx + 8dy\\ z - (-14) &= -10(x-1) + 8(y-2)\\ z &= -10x + 8y + 10 - 16 - 14 = -10x + 8y - 20\end{aligned}$$

while the tangent plane at $(2,1)$ is given by

$$\begin{aligned}dz &= \frac{\partial z}{\partial x}_{|x=2,y=1} dx + \frac{\partial z}{\partial y}_{|x=2,y=1} dy = -15dx - 8dy\\ z - 0 &= -15(x-2) - 8(y-1)\\ z &= -15x - 8y + 30 + 8 = -15x - 8y + 38\end{aligned}$$

Exercise 5 For the function

$$z = xy$$

find the slope of the level curves in the points $(1,2)$ and $(2,2)$. Find the equations for the tangents.

Solution 5 The equation for the tangent to a level curves is given by

$$0 = dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = ydx + xdy$$

The tangent to the level curves at $(1,2)$ is hence given by

$$\begin{aligned}0 &= dz = \frac{\partial z}{\partial x}_{|x=1,y=2} dx + \frac{\partial z}{\partial y}_{|x=1,y=2} dy = 2dx + dy\\ 0 &= 2(x-1) + (y-2) \Leftrightarrow y = 4 - 2x\end{aligned}$$

and has slope -2. The tangent to the level curves at $(2,2)$ is hence given by

$$\begin{aligned}0 &= dz = \frac{\partial z}{\partial x}_{|x=2,y=2} dx + \frac{\partial z}{\partial y}_{|x=2,y=2} dy = 2dx + 2dy\\ 0 &= 2(x-2) + 2(y-2) \Leftrightarrow y = 4 - x\end{aligned}$$

and has slope -1.

Exercise 6 Suppose a consumer has the utility function $u(x,y) = xy$ and his budget is 40. Given your knowledge of economics from economics and the results from the previous question, what will be his demand when the prices are a) $p_x = p_y = 10$ and b) $p_x = 20$, $p_y = 10$? (Error in original questions, where p_x was given as 40 in previous question.)

Solution 6 The optimum is characterized by the condition that at it the budget line is tangent to indifference curve. The budget line given by the equation $10x + 10y = 40$ or $y = 4 - x$ has slope -1. The consumption bundle $(2, 2)$ costs exactly 40 at these prices and is hence on the budget line. The slope of the (tangent to the) indifference curve at this point is -1. Hence $(2, 2)$ is the optimal consumption bundle for these prices and budget. Similarly, the budget line $20x + 10y = 40$ or $y = 4 - 2x$ has slope -2 and goes through the point $(1, 2)$. From the previous question we know that the indifference curve is here tangent to the budget line and so $(1, 2)$ is the optimal consumption bundle.

Exercise 7 Find the critical point of the function

$$z = 20x^2 - 37xy + 31x + 15y^2 - 16y - 7$$

Is it a maximum, a minimum or a saddle point?

Solution 7

$$\begin{aligned}\frac{\partial z}{\partial x} &= 40x - 37y + 31 \\ \frac{\partial z}{\partial y} &= -37x + 30y - 16\end{aligned}$$

The solution to

$$\begin{aligned}40x - 37y + 31 &= 0 \\ -37x + 30y - 16 &= 0\end{aligned}$$

is $x = 2, y = 3$. This is the critical point. The Hessian matrix is

$$\begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 40 & -37 \\ -37 & 30 \end{bmatrix}$$

We have

$$\det \begin{bmatrix} 40 & -37 \\ -37 & 30 \end{bmatrix} = 1200 - (40 - 3)^2 = 1200 - 1600 + 240 - 9 = -169 < 0$$

The critical point is hence a saddle point.

Exercise 8 A T-shirt shop carries two competing shirts, one endorsed by Michael Jordan and the other by Shaq O'Neal. The owner of the store can obtain both at a cost of \$2 per shirt and estimates that if Jordan shirts are sold for x dollars apiece and O'Neal shirts for y dollars apiece, consumers will buy approximately $40 - 50x + 40y$ Jordan shirts and $20 + 60x - 70y$ O'Neil shirts each day.

a) Express as functions of x and y: i) the revenue from selling Jordan shirts, ii) the revenue from selling O'Neal shirts iii) the costs for shirts and iv) the overall profit.

b) Find the critical point of the profit function. Show that it is the unique maximum.

Solution 8 The profit function is

$$\Pi(x,y) = (x-2)(40-50x+40y) + (y-2)(20+60x-70y)$$

We have (using the product rule)

$$\begin{aligned}
\frac{\partial \Pi}{\partial x} &= 1 \times (40-50x+40y) + (x-2) \times (-50) + (y-2) \times 60 = -100x + 100y + 20 \\
\frac{\partial \Pi}{\partial y} &= (x-2) \times 40 + 1 \times (20+60x-70y) + (y-2) \times (-70) = 100x - 140y + 80
\end{aligned}$$

and obtain the system of simultaneous equations

$$\begin{aligned}
-100x + 100y + 20 &= 0 \\
100x - 140y + 80 &= 0
\end{aligned}$$

which yields the critical point $x = 2.7$, $y = 2.5$. The Hessian matrix is

$$H = \begin{bmatrix} \frac{\partial^2 \Pi}{\partial x^2} & \frac{\partial^2 \Pi}{\partial y \partial x} \\ \frac{\partial^2 \Pi}{\partial x \partial y} & \frac{\partial^2 \Pi}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -100 & 100 \\ 100 & -140 \end{bmatrix}$$

We have $\frac{\partial^2 \Pi}{\partial x^2} = -100 < 0$ and

$$\begin{aligned}
\det H &= (-100) \times (-140) - 100 \times 100 \\
&= 100 \times 140 - 100 \times 100 = 100 \times (140 - 100) > 0
\end{aligned}$$

so the function is strictly concave and the critical point hence a (global) profit maximum.