Exercise 1 Bring onto common denominator and simplify

\[ \frac{1}{x-2} - \frac{1}{x+2} \]

Solution 1

\[ \frac{1}{x-2} - \frac{1}{x+2} = \frac{(x+2) + (x-2)}{(x-2)(x+2)} = \frac{2x}{x^2 - 4} \]

Exercise 2 Bring onto common denominator and simplify

\[ \frac{1}{x^2} - \frac{1}{y^2} \]

Solution 2

\[ \frac{1}{x^2} - \frac{1}{y^2} = \frac{y^2 - x^2}{x^2y^2} = \frac{x^2y^2}{y^2x^2} \times \frac{x^2y^2}{y^2x^2} = \frac{y^2 - x^2}{y^2 + x^2} \]

Exercise 3 Simplify

\[ \frac{\sqrt[3]{aa^{1/12}} \sqrt[6]{a^3}}{a^{\frac{5}{12}} \sqrt{a}} \]

Solution 3

\[ \frac{\sqrt[3]{aa^{1/12}} \sqrt[6]{a^3}}{a^{\frac{5}{12}} \sqrt{a}} = a^{\frac{1}{3}} a^{\frac{1}{12}} a^{\frac{3}{6}} a^{-\frac{5}{12}} a^{-\frac{1}{2}} = a^{\frac{1}{3} + \frac{1}{12} + \frac{3}{6} - \frac{5}{12} - \frac{1}{2}} = a^{\frac{4+1+9-5-6}{12}} = a^{\frac{14-5\sqrt{6}}{12}} = a^{\frac{14-5\sqrt{6}}{12}} = \sqrt[12]{a^{14-5\sqrt{6}}} \]

Notice there was a typo in the original question. With that typo we would get

\[ \frac{\sqrt[3]{aa^{1/12}} \sqrt[6]{a^3}}{a^{\frac{5}{12}} \sqrt{a}} = a^{\frac{1}{3}} a^{\frac{1}{12}} a^{\frac{3}{6}} a^{-\frac{5}{12}} a^{-\frac{1}{2}} = a^{\frac{1}{3} + \frac{1}{12} + \frac{3}{6} - \frac{5}{12} - \frac{1}{2}} = a^{\frac{6+1+9-5\sqrt{6}}{12}} = a^{\frac{14-5\sqrt{6}}{12}} = \frac{12}{a^{14-5\sqrt{6}}} \]

Exercise 4 Solve

\[ \frac{x-2}{x+3} = \frac{x-4}{x+4} \]

Solution 4

\[ (x-2)(x+4) = (x-4)(x+3) \]
\[ x^2 - 2x + 4x - 8 = x^2 - 4x + 3x - 12 \]
\[ +2x - 8 = -x - 12 \]
\[ 4 = -3x \]
\[ x = \frac{-4}{3} = -\frac{1}{3} \]

\[ x = -\frac{4}{3} = -\frac{1}{3} \]
Exercise 5 Solve
\[
\frac{1}{4} K^{-1/2} L^{1/4} = \frac{r}{w} \quad \text{for } L
\]

Solution 5 We first simplify
\[
\frac{1}{4} K^{-1/2} L^{1/4} = \left( \frac{1}{2} \times \frac{4}{1} \right) K^{-1/2} L^{1/4} L^{3/4} K^{-1/2} = 2 K^{-1/2} L^{1/4+3/4} = 2 K^{-1} L^1 = 2 \frac{L}{K}
\]
and so
\[
2 \frac{L}{K} = \frac{r}{w} \quad \text{and so} \quad L = \frac{1}{2} w K
\]

Exercise 6 Simplify
\[
a - a \sum_{t=0}^{T} \beta^t
\]

Solution 6
\[
a - a \sum_{t=0}^{T} \beta^t = a \left( 1 - \frac{1}{\sum_{t=0}^{T} \beta^t} \right) = a \left( \frac{\sum_{t=0}^{T} \beta^t - 1}{\sum_{t=0}^{T} \beta^t} \right)
\]

or, more explicitly
\[
a \left( \frac{\sum_{t=0}^{T} \beta^t - 1}{\sum_{t=0}^{T} \beta^t} \right) = a \frac{1 + \beta + \beta^2 + \ldots + \beta^t}{1 + \beta + \beta^2 + \ldots + \beta^T} = a \frac{\beta \sum_{t=0}^{T-1} \beta^t}{\sum_{t=0}^{T} \beta^t}
\]

Exercise 7 Simplify
\[
\ln \left( x^4 \exp (-x) \right)
\exp \left( \ln \left( x^2 \right) - 2 \ln y \right)
\]

Solution 7
\[
\ln \left( x^4 \exp (-x) \right) = \ln \left( x^4 \right) + \ln \left( \exp (-x) \right) = 4 \ln x - x
\exp \left( \ln \left( x^2 \right) - 2 \ln y \right) = \exp \left( \ln \left( x^2 \right) - \ln \left( y^2 \right) \right) = \exp \left( \ln \frac{x^2}{y^2} \right) = \frac{x^2}{y^2}
\]

Exercise 8 Calculate the first and the second derivative of the function
\[
y = \sqrt[3]{1 - x^3}
\]
Solution 8

\[ y = (1 - x^3)^{\frac{1}{3}} \]

\[ \frac{dy}{dx} = \frac{1}{3} (1 - x^3)\left(\frac{1}{3} - 1\right) \frac{d(1 - x^3)}{dx} = \frac{1}{3} (1 - x^3)^{-\frac{2}{3}} \times (-3x^2) \]

\[ = -\frac{x^2}{\sqrt[3]{(1 - x^3)^2}} \]

Exercise 9 Find the peaks and the troughs of the function

\[ y = x^3 - 10.5x^2 + 30x + 20 \]

Solution 9

\[ \frac{dy}{dx} = 3x^2 - 21x + 30 = 0 \Leftrightarrow x^2 - 7x + 10 = (x - 2)(x - 5) = 0 \]

The function has critical points at \( x = 2 \) and \( x = 10 \).

\[ \frac{d^2y}{dx^2} = 6x - 21 \]

The function has a negative second derivative and is hence concave for \( x < 7/2 \). The function has a positive second derivative and is hence convex for \( x > 7/2 \). Therefore \( x = 2 \) is a peak and \( x = 5 \) a trough.

Exercise 10 Find the absolute maximum and minimum of the function

\[ y(x) = x^3 - 10.5x^2 + 30x + 20 \]

1) on the interval \( 1 \leq x \leq 6 \)
2) on the interval \( 1 \leq x \leq 3 \)
3) on the interval \( 3 \leq x \leq 4 \)

Solution 10 The function has critical points at \( x = 2 \) and \( x = 7 \) as we saw in the previous question. Moreover,

\[ y(2) = 8 - 42 + 60 + 20 = 46 \]
\[ y(5) = 125 - 250 - 12.5 + 150 + 20 = 32.5 \]

1) Since

\[ y(1) = 1 - 10.5 + 30 + 20 = 40.5 \]
\[ y(6) = 216 - 378 + 180 + 20 = 38 \]

the absolute minimum and maximum on the interval \([0, 6]\) are at \( x = 5 \) and at \( x = 2 \).

2) Since

\[ y(1) = 1 - 10.5 + 30 + 20 = 40.5 \]
\[ y(3) = 27 - 90 - 4.5 + 90 + 20 = 42.5 \]
the absolute minimum and maximum on the interval \([0, 3]\) are at \(x = 1\) and at \(x = 2\).

3) Since

\[
y(3) = 27 - 90 - 4.5 + 90 + 20 = 42.5 \\
y(4) = 64 - 160 - 8 + 120 + 20 = 36
\]

the absolute minimum and maximum on the interval \([3, 4]\) are at \(x = 4\) and at \(x = 3\).

Exercise 11 A bus company will charter a bus that holds 50 people to groups of 35 or more. If a group contains exactly 35 people, each person pays £60. In large groups, everybody’s fare is reduced by £1 for each person in excess of 35. Determine the size of the group for which the bus company’s revenue will be greatest.

Solution 11 Let \(x\) be the number of people in excess of 35. The price per person is then \(60 - x\). Since there are \(35 + x\) people on the bus, the revenue is

\[
R(x) = (35 + x)(60 - x)
\]

This quadratic function with leading term \(-x^2\) has roots at \(x = -35\) and \(x = 60T\). It is maximized in between the two roots at \(-\frac{35+60}{2} = \frac{25}{2}\). Revenue is hence maximized at \(35 + 12.5 = 46.5\) people, with a fare of £60 - £12.5 = £47.5 and a revenue of \(R(12.5) = 2256.25\). Given that there are no half people, revenue is maximized if there are 12 or 13 people in excess of 35 and the price is 48 or 47. Revenues in these two cases are \(R(12) = R(13) = 2256\).

Instead of using the properties of quadratic functions as above we can differentiate and get by the product rule

\[
\frac{dR}{dx} = 60 - x - (35 + x) = 25 - 2x \\
\frac{d^2R}{dx^2} = -2 < 0
\]

so we have a strictly concave function with a maximum at \(x = 25/2\).

And, still alternatively, \(R\) has the derivative \(25 - 2x\). It has hence a critical point at 12.5. To the left the derivative is positive, to the right negative. this means that the function is increasing to the left, decreasing to the right. Thus 12.5 is a global maximum.