BEEM103
January 2010
OPTIMIZATION TECHNIQUES FOR ECONOMISTS
solutions

Part A (You can gain *no more than 55 marks* on this part.)

Problem 1 (10 marks) Simplify

$$\frac{1}{8}\frac{\sqrt{x^2y^{-1}}}{x^3} - \frac{2}{3}\frac{1}{x^2\sqrt{y}} \qquad \text{and} \qquad \frac{x/\sqrt[3]{x^2}}{x^{\frac{7}{3}}}$$

Solution 1

$$\begin{aligned} \frac{1}{8}\frac{\sqrt{x^2y^{-1}}}{x^3} - \frac{2}{3}\frac{1}{x^2\sqrt{y}} &= \frac{3}{24}\frac{1}{x^2\sqrt{y}} - \frac{16}{24}\frac{1}{x^2\sqrt{y}} = -\frac{13}{24}\frac{1}{x^2\sqrt{y}} \\ \frac{x/\sqrt[3]{x^2}}{x^{\frac{5}{3}}} &= \frac{x/x^{\frac{2}{3}}}{x^{\frac{7}{3}}} = \frac{x^{\frac{1}{3}}}{x^{\frac{7}{3}}} = x^{\frac{1}{3}-\frac{7}{3}} = x^{-\frac{6}{3}} = x^{-2} = \frac{1}{x^2} \end{aligned}$$

Problem 2 (10 marks) Solve

$$x^{3-\ln x} = x^{2+\ln x}$$

Solution 2

$$\begin{aligned} \ln x^{3-\ln x} &= (3-\ln x)\ln x \\ \ln x^{2+\ln x} &= (2+\ln x)\ln x \\ (3-\ln x)\ln x &= (2+\ln x)\ln x \\ (-1+2\ln x)\ln x &= 0 \end{aligned}$$

Solutions: $\ln x = 0$, ie. $x = 1$, or $\ln x = \frac{1}{2}$ or $x = e^{\frac{1}{2}} = \sqrt{e}$

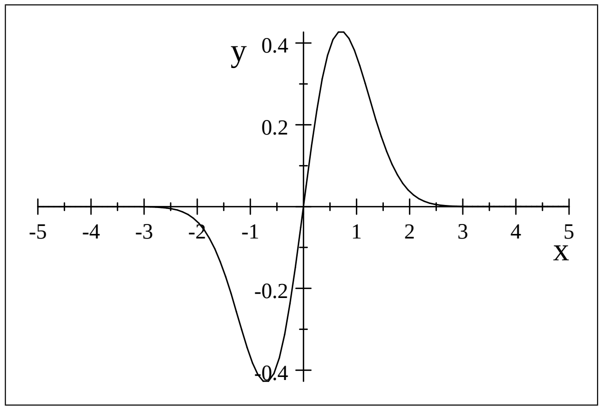
Problem 3 (10 marks) Consider the function

$$y(x) = xe^{-\frac{x^2}{2}}$$

i) Calculate and draw a sign diagram for the first derivative. Where is the function increasing or decreasing? Are there any peaks or troughs? Is the function quasi concave on the interval of positive numbers $x > 0$? Does the function have a global maximum over the positive numbers $x > 0$?

ii) Calculate and draw a sign diagram for the second derivative. Where is the function convex or concave? Are there any inflection points?

Solution 3



a) Using the product and chain rule we obtain

$$y' = (x)' e^{-\frac{x^2}{2}} + x \left(e^{-\frac{x^2}{2}}\right)' = \left(1 - x^2\right) e^{-\frac{x^2}{2}}$$

Since $e^{-\frac{x^2}{2}}$ ia always positive there are critical points at -1 and $+1$. Because $1 - x^2$ is positive inside $(-1, 1)$ and negative outside $[-1, 1]$, the function is increasing inside and decreasing outside the interval $[-1, +1]$. Thus $+1$ is a peak, -1 a trough. On the set of positive numbers the function is increasing until it reaches a global maximum at $+1$ and is then decreasing. In particular, it is quasiconcave on the positive numbers.

b)

$$\begin{aligned} y'' &= (-2x) e^{-\frac{x^2}{2}} - 2x \left(1 - x^2\right) e^{-\frac{x^2}{2}} \\ &= -2x \left(2 - x^2\right) e^{-\frac{x^2}{2}} \end{aligned}$$

The second derivative is zero at $x = 0, \pm\sqrt{2}$. It is non-negative (and the function hence convex) on the intervals $\left[-\sqrt{2}, 0\right]$ and $\left[\sqrt{2}, +\infty\right)$. It is non-positive (and the function hence concave) on the remaining intervals.

Problem 4 (10 marks) For the function

$$y = x^4 - 2x^2$$

find the (global) maxima and minima a) on the interval $[-1, 1]$ and b) on the interval $[0, 2]$.

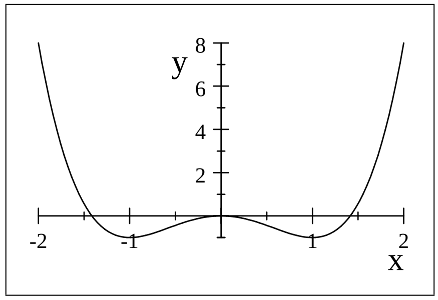
Solution 4

$$y' = 4x^3 - 4x$$

The derivative is zero when $x = 0$ or $x = \pm 1$. We have $y(\pm 1) = -1$, $y(0) = 0$.

On the interval $[-1, 1]$ the maximum is hence at $x = 0$ while $x = 1$ and $x = -1$ are the two minima. Since $y(2) = 8$ the maximum on $[0, 2]$ is at $x = 2$ while the minimum is at $x = 1$.

This is illustrated by the graph of the function:



Problem 5 (10 marks) Find the equation of the tangent plane of

$$z(x,y) = \ln(5x+y) + \ln(10x+y)$$

at the point $(x^*, y^*, z^*) = (2, 3, z(2,3))$.

Solution 5

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{5}{5x+y} + \frac{10}{10x+y} \\
\frac{\partial z}{\partial x}_{|(2,3)} &= \frac{5}{10+3} + \frac{10}{20+3} = \frac{5}{13} + \frac{10}{23} = \frac{245}{299} \\
\frac{\partial z}{\partial y} &= \frac{1}{5x+y} + \frac{1}{10x+y} \\
\frac{\partial z}{\partial y}_{|(2,3)} &= \frac{1}{10+3} + \frac{1}{20+3} = \frac{1}{13} + \frac{1}{23} = \frac{36}{299} \\
z(2,3) &= \ln 13 + \ln 23
\end{aligned}$$

The equation for the tangent is hence

$$\begin{aligned}
z &= \left(\frac{5}{13} + \frac{10}{23}\right)(x-2) + \left(\frac{1}{13} + \frac{1}{23}\right)(y-3) - \ln 13 - \ln 23 \\
&= \left(\frac{5}{13} + \frac{10}{23}\right)x + \left(\frac{1}{13} + \frac{1}{23}\right)y - \ln 13 - \ln 17 - 2\left(\frac{5}{13} + \frac{10}{23}\right) - 3\left(\frac{1}{13} + \frac{1}{23}\right) \\
&= \frac{245}{299}x + \frac{36}{299}y - \ln 13 - \ln 17 - 2
\end{aligned}$$

Problem 6 The only grocery store in a small rural community carries two brands of frozen apple juice, a local brand that it obtains at the cost of 30 cent per can and a well-known national brand that it obtains at a cost of 40 cent a can. The grocer estimates that if the local brand is sold for x cents per can and the national brand for y cents per can, approximately $(70 - 5x + 4y)$ cans of the local brand and $(80 + 6x - 7y)$ cans of the national brand will be sold each day. How should the grocer price each brand to maximize the profit from the sale of the juice?

Solution 6 The profit function is

$$\Pi(x,y) = (x-30)(70-5x+4y) + (y-40)(80+6x-7y)$$

We have (using the product rule)

$$\begin{aligned}\frac{\partial \Pi}{\partial x} &= 1\times(70-5x+4y)+(x-30)\times(-5)+(y-40)\times 6 = 10y-10x-20 \\ \frac{\partial \Pi}{\partial y} &= (x-30)\times 4 + 1\times(80+6x-7y)+(y-40)\times(-7) = 10x-14y+240\end{aligned}$$

$10x - 14y + 240$ and obtain the system of simultaneous equations

$$\begin{aligned}-10x+10y-20 &= 0 \\ 10x-14y+240 &= 0\end{aligned}$$

Addition yields $-4y+220=0$ or $y=55$. From the first equation $x=y-2=53$. Thus $(x^*,y^*)=(53,55)$ is the unique critical point of the function. The Hessian matrix is

$$H = \begin{bmatrix} \frac{\partial^2\Pi}{\partial x^2} & \frac{\partial^2\Pi}{\partial y\partial x} \\ \frac{\partial^2\Pi}{\partial x\partial y} & \frac{\partial^2\Pi}{\partial y^2}\end{bmatrix} = \begin{bmatrix} -10 & 10 \\ 10 & -14\end{bmatrix}$$

We have $\frac{\partial^2\Pi}{\partial x^2} = -10 < 0$ and

$$\begin{aligned}\det H &= (-10)\times(-14) - 10\times 10 = 40 > 0 \\ &= 100\times 140 - 100\times 100 = 100\times(140-100) > 0\end{aligned}$$

so the function is strictly concave and the critical point hence a (global) profit maximum.

Problem 7 (10 marks) Find a solution to the differential equation

$$\frac{dx}{dt} = t^4 x^2$$

Solution 7

$$\begin{aligned} x^{-2}dx &= t^4 dt \\ \int x^{-2}dx &= \int t^4 dt \\ -x^{-1} &= \frac{1}{5}t^5 + C \\ x &= -\frac{5}{t^5+C}\end{aligned}$$

Test: $\frac{dx}{dt} = -\left(-5\left(t^5+C\right)^{-2}\times 5t^4\right) = 25\frac{t^4}{(t^5+C)^2}$, $t^4x^2 = t^4\frac{25}{(t^5+C)^2}$

Problem 8 (10 marks) Solve the problem

$$\min_{x(t)} \int_0^1 \left(1+x+x^2+\dot{x}+\dot{x}^2\right)e^{rt}dt$$

subject to the restrictions $x(0)=0$, $x(1)=1$.

Solution 8

$$\begin{aligned} F(t,x,\dot{x}) &= \left(1+x+x^2+\dot{x}+\dot{x}^2\right)e^{\frac{3}{2}t} \\ \frac{\partial F}{\partial x} &= (1+2x)\,e^{\frac{3}{2}t} \\ \frac{\partial F}{\partial \dot{x}} &= (1+2\dot{x})\,e^{\frac{3}{2}t} \\ \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} &= \left(\frac{3}{2}+3\dot{x}+2\ddot{x}\right)e^{\frac{3}{2}t} \end{aligned}$$

The Euler equation becomes

$$\begin{aligned} (1+2x)\,e^{\frac{3}{2}t} &= \left(\frac{3}{2}+3\dot{x}+2\ddot{x}\right)e^{\frac{3}{2}t} \\ 1+2x &= \frac{3}{2}+3\dot{x}+2\ddot{x} \\ -\frac{1}{2} &= 2\ddot{x}+3\dot{x}-2x \\ -\frac{1}{4} &= \ddot{x}+\frac{3}{2}\dot{x}-x \end{aligned}$$

A special solution to this differential equation is $x(t) = -\frac{1}{4}$, but this would not satisfy the boundary conditions. The characteristic polynomial for the homogeneous DE $\ddot{x}+\frac{3}{2}\dot{x}-x = 0$ is

$$0 = \alpha^2 + \frac{3}{2}\alpha - 1$$

which has the roots $\frac{1}{2}, -2$. The general solution to the Euler equation is hence

$$x(t) = -\frac{1}{4} + Ae^{\frac{1}{2}t} + Be^{-2t}$$

The boundary conditions are now

$$\begin{aligned} 0 &= -\frac{1}{4} + A + B \\ 1 &= -\frac{1}{4} + Ae^{\frac{1}{2}} + Be^{-2} \end{aligned}$$

which have the solution

$$A = \frac{e^{-2}-5}{4e^{-2}-4e^{\frac{1}{2}}},\, B = -\frac{e^{\frac{1}{2}}-5}{4e^{-2}-4e^{\frac{1}{2}}}$$

Our unique candidate for an optimum is thus

$$x(t) = -\frac{1}{4} + \frac{e^{-2}-5}{4e^{-2}-4e^{\frac{1}{2}}}e^{\frac{1}{2}t} - \frac{e^{\frac{1}{2}}-5}{4e^{-2}-4e^{\frac{1}{2}}}e^{-2t}$$

and this is indeed a minimum because $-F$ is concave in both arguments.

Part B (You can gain *no more than 15 marks* on this part.)

Problem 9 (15 marks) Determine the optimal profit for a firm with the production function

$$Q = A \ln K + B \ln L$$

$(A, B > 0)$ operating in perfectly competitive input- and output markets and facing the output price $P > 0$, the interest rate $r > 0$ and the wage rate w.

Solution 9

$$\Pi = PQ - rK - wL$$

$$\begin{aligned} \frac{\partial \Pi}{\partial K} &= AP\frac{1}{K} - r = 0 \\ \frac{\partial \Pi}{\partial K} &= BP\frac{1}{L} - w = 0 \end{aligned}$$

Division yields

$$\begin{aligned} \frac{AL}{BK} &= \frac{r}{w} \\ K &= \frac{A}{B}\frac{w}{r}L \end{aligned}$$

Plugging this into the first equation above yields

$$AP\frac{B}{A}\frac{r}{w}\frac{1}{L} = r \Leftrightarrow \frac{PB}{wL} = 1 \Leftrightarrow L = \frac{BP}{w}$$

and symmetrically we obtain

$$K = \frac{AP}{r}$$

The solution is economically meaningful because both K and L are positive. To see that we indeed have a maximum we calculate the Hessian matrix is

$$H = \begin{bmatrix} -APK^{-2} & 0 \\ 0 & -BPL^{-2} \end{bmatrix}$$

which has a negative top-left entry and a positive determinant

$$\det H = ABP^2K^{-2}L^{-2} > 0$$

The profit function is thus convex and the critical point a global maximum.

Problem 10 (15 marks) For the point $(1, -3)$ find the point (x^*, y^*) that minimizes the distance

$$\sqrt{(x-a)^2 + (y-b)^2}$$

subject to the constraints

$$\begin{aligned} -1 &\leq x \leq 1 \\ x - 1 &\leq y \leq x + 1. \end{aligned}$$

Solution 10 A graph indicates that only the constraint $y = x - 1$ is binding in the optimum. We start hence with the Lagrangian

$$\mathcal{L} = -(x-1)^2 - (y+3)^2 + \lambda(1 - x + y)$$

asuuming the the Lagrangian multipliers for the other constraints are all zero. The partial derivatives are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= -2(x-1) - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= -2(y+3) + \lambda = 0 \end{aligned}$$

Eliminating λ yields $-(x-1) = y + 3$, $y = -x - 2$. Since also $y = x - 1$ is supposed to hold we get $-x - 2 = x - 1$, $2x = 1$ or $x = \frac{1}{2}$. This implies $y = -\frac{1}{2}$. The solution to the first order-conditions an complementarity conditions yield $(x^*, y^*) = \left(\frac{1}{2}, -\frac{1}{2}\right)$, which satisfies all constraints. We have $\lambda = 2(y^* + 3) = 2 \times 2.5 > 0$. Given that $\mathcal{L}$ is concave in x and y it follows that $(x^*, y^*) = \left(\frac{1}{2}, -\frac{1}{2}\right)$ is the constrained optimum of our problem.

Problem 11 (15 marks) Solve the problem

$$\max_{u(t)} \int_0^1 -(x-5)^2 \, dt$$

subject to $\dot{x} = u$, $x(0) = 0$, $x(1) = 2$, $-1 \leq u \leq 3$

Solution 11 The Hamiltonian is

$$H = -(x-5)^2 + pu$$

To maximize H we must have

$$u(t) = \begin{cases} 5 \text{ if } p(t) > 0 \\ -1 \text{ if } p(t) < 0 \end{cases}$$

The differential equation for $p(t)$ is

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = -2(x(t) - 5)$$

Because $\dot{x}(t) = u(t) \leq 5$ and $x(0) = 0$ we have $x(t) < 5$ for all $t > 1$. Hence $p(t)$ is strictly decreasing. This implies that $u(t)$ can only change sign once, from being positive and have value $+5$ to being negative and having value -5.

Given the boundary conditions, this only leaves one candidate for $x(t)$, namely

$$x(t) = \begin{cases} 5t \text{ if } t < t_0 \\ 3 - t \text{ if } t \geq t_0 \end{cases}$$

where t_0 is chosen such that $x(t)$ is continuous at t_0, so $5t = 3 - t$, which means that $t_0 = \frac{1}{2}$.

We have hence $p\left(\frac{1}{2}\right) = 0$. For $t < t_0$

$$p(t) = \int_t^{\frac{1}{2}} 5\tau d\tau = \frac{5}{8} - \frac{5}{2}t^2$$

and for $t > t_0$

$$p(t) = \int_{\frac{1}{2}}^{t} (3-\tau)\, d\tau = -\frac{1}{2}t^2 + 3t - \frac{11}{8}$$

given that the objective function is concave, this is indeed the optimum.

Part C

Problem 12 (20 marks) Derive the demand function of a consumer with utility function

$$u(x,y) = \sqrt{x} + \sqrt{y}$$

Solution 12 The utility function is monotonic and so the budget constraint must be binding. The Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \sqrt{x} + \sqrt{y} + \lambda_1 x + \lambda_2 y + \lambda_3 (b - p_x x - p_y y) \\
\frac{\partial \mathcal{L}}{\partial x} &= \frac{1}{2\sqrt{x}} + \lambda_1 - \lambda_3 p_x = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= \frac{1}{2\sqrt{y}} + \lambda_2 - \lambda_3 p_y = 0
\end{aligned}$$

Suppose first that only the budget constraint binds and so $\lambda_1 = \lambda_2 = 0$. In this case

$$\begin{aligned}
\frac{1}{2\sqrt{x}} &= \lambda_3 p_x \\
\frac{1}{2\sqrt{y}} &= \lambda_3 p_y \\
\Rightarrow \frac{\sqrt{y}}{\sqrt{x}} &= \frac{p_x}{p_y} \\
\sqrt{y} &= \frac{p_x}{p_y}\sqrt{x} \\
y &= \frac{p_x^2}{p_y^2} x
\end{aligned}$$

Next we use the budget equation.

$$\begin{aligned}
p_x x + p_x \frac{p_x^2}{p_y^2} x &= b \\
p_x \left(\frac{p_x^2 + p_y^2}{p_y^2}\right) x &= b \\
x^* &= \frac{p_y^2}{p_x^2 + p_y^2} \frac{b}{p_x} > 0 \\
y^* &= \frac{p_x^2}{p_x^2 + p_y^2} \frac{b}{p_y} > 0
\end{aligned}$$

This solution is admissible for any $p_x, p_y, b > 0$. The Lagrangian multiplier λ_3 for the budget constraint is positive

$$\lambda_3 = \frac{1}{2p_x\sqrt{x^*}} > 0.$$

We do not have to consider any further cases, which is of course a bit too simple for a proper exam question in part C! Because the utility function is easily seen to be concave and all constraints linear, we have found the optimum for all $p_x, p_y, b > 0$. Thus we have determined the demand function for both commodities. The case where Lagrangian

e must have

$$-b \leq p_x - p_y \leq b$$

Next, if the non-negativity constraint $x = 0$ is binding we get the optimum $x^* = 0$ and $y^* = b/p_y$. For the Lagrange multipliers we obtain $\lambda_2 = 0$,

$$\begin{aligned} \frac{1}{b/p_y + 1} &= \lambda_3 p_y \\ \lambda_3 &= \frac{1}{b + p_y} \\ \lambda_1 &= \lambda_3 p_x - 1 = \frac{p_x}{b + p_y} - 1 \geq 0 \\ &\iff p_x - p_y \geq b \end{aligned}$$

This case arises when $p_x - p_y \geq b$.

Finally, if the non-negativity constraint $y = 0$ is binding, we must have $y^* = 0$ and $x^* = b/p_x$. Since $\lambda_1 = 0$ we obtain

$$\begin{aligned} \frac{1}{b/p_x + 1} &= \lambda_3 p_x \\ \lambda_3 &= \frac{1}{b + p_x} \\ \lambda_2 &= \lambda_3 p_y - 1 = \frac{p_y}{b + p_x} - 1 \geq 0 \\ &\iff p_x - p_y \leq -b \end{aligned}$$

Overall we obtain the demand function

$$(x^*(p_x, p_y, b), y^*(p_x, p_y, b)) = \begin{cases} (p/b_x, 0) & \text{for} \quad p_x - p_y \leq -b \\ \left(\frac{b - p_x + p_y}{2p_x}, \frac{b - p_y + p_x}{2p_y}\right) & \text{for} \quad -b \leq p_x - p_y \leq b \\ (0, p/b_y) & \text{for} \quad b \leq p_x - p_y \end{cases}$$

Problem 13 (20 marks) Solve the production planning problem

$$\min \int_0^T \left(c_1 u^2 + c_2 x\right) dt$$

subject to $\dot{x} = u$, $x(0) = 0$, $x(T) = B$, $u(t) \geq 0$ in the case $B < c_2 T^2 / 4c_1$, taking explicit account of the constraint $u \geq 0$.

Solution 13 see Kamine Schwartz, p. 172f.