BEEM103

UNIVERSITY OF EXETER

BUSINESS SCHOOL

January 2010

test exam

OPTIMIZATION TECHNIQUES FOR ECONOMISTS

Module convenor: Dieter Balkenborg

Duration : **TWO HOURS**

The paper has 3 parts. Your marks on the *first part* will be *rounded down* to *55 marks*. Your marks on the *second part* will be *rounded down* to *15 marks.* Obviously, you cannot get over 100 marks overall. This examination is closed book and no materials are allowed. **Full work must be shown** on your script. **Please write legibly.**

Part A (You can gain *no more than 55 marks* on this part.)

Problem 1 (10 marks) Simplify

$$\frac{1}{8}\frac{\sqrt{x^2y^{-1}}}{x^3} - \frac{2}{3}\frac{1}{x^2\sqrt{y}} \qquad \text{and} \qquad \frac{x/\sqrt[3]{x^2}}{x^{\frac{7}{3}}}$$

Problem 2 (10 marks) Solve

$$x^{3-\ln x} = x^{2+\ln x}$$

Problem 3 (10 marks) Consider the function

$$y(x) = xe^{-\frac{x^2}{2}}$$

i) Calculate and draw a sign diagram for the first derivative. Where is the function increasing or decreasing? Are there any peaks or troughs? Is the function quasi concave on the interval of positive numbers $x > 0$? Does the function have a global maximum over the positive numbers $x > 0$?

ii) Calculate and draw a sign diagram for the second derivative. Where is the function convex or concave? Are there any inflection points? *Hint:* The second derivative can be written in the form $\frac{2xq(x)}{(x^2+1)^3}$ where $q(x)$ is a quadratic function.

Problem 4 (10 marks) For the function

$$y = x^4 - 2x^2$$

find the (global) maxima and minima a) on the interval $[-1, 1]$ and b) on the interval $[0, 2]$.

Problem 5 (10 marks) Find the equation of the tangent plane of

$$z(x, y) = \ln(5x + y) + \ln(10x + y)$$

at the point $(x^*, y^*, z^*) = (2, 3, z(2, 3))$.

Problem 6 (10 marks) The only grocery store in a small rural community carries two brands of frozen apple juice, a local brand that it obtains at the cost of 30 cent per can and a well-known national brand that it obtains at a cost of 40 cent a can. The grocer estimates that if the local brand is sold for x cents per can and the national brand for y cents per can, approximately $(70 - 5x + 4y)$ cans of the local brand and $(80 + 6x - 7y)$ cans of the national brand will be sold each day. How should the grocer price each brand to maximize the profit from the sale of the juice?

Problem 7 (10 marks) Find a solution to the differential equation

$$\frac{dx}{dt} = t^4 x^2$$

Problem 8 (10 marks) Solve the problem

$$\min_{x(t)} \int_0^1 \left(1 + x + x^2 + \dot{x} + \dot{x}^2\right) e^{rt} dt$$

subject to the restrictions $x(0) = 0$, $x(1) = 1$.

Part B (You can gain *no more than 15 marks* on this part.)

Problem 9 (15 marks) Determine the optimal profit for a firm with the production function

$$Q = A \ln K + B \ln L$$

$(A, B > 0)$ operating in perfectly competitive input- and output markets and facing the output price $P > 0$, the interest rate $r > 0$ and the wage rate w.

Problem 10 (15 marks) For the point $(1, -3)$ find the point (x^*, y^*) that minimizes the distance

$$\sqrt{(x-a)^2 + (y-b)^2}$$

subject to the constraints

$$\begin{aligned} -1 &\leq x \leq 1 \\ x - 1 &\leq y \leq x + 1. \end{aligned}$$

Problem 11 (15 marks) Solve the problem

$$\max_{u(t)} \int_0^1 -(x-5)^2 \, dt$$

subject to $\dot{x} = u$, $x(0) = 0$, $x(1) = 2$, $-1 \leq u \leq 3$.

Part C

Problem 12 (20 marks) Derive the demand function of a consumer with utility function

$$u(x, y) = \sqrt{x} + \sqrt{y}$$

Problem 13 (20 marks) Solve the production planning problem

$$\min \int_0^T \left(c_1 u^2 + c_2 x\right) dt$$

subject to $\dot{x} = u$, $x(0) = 0$, $x(T) = B$, $u(t) \geq 0$ in the case $B < c_2 T^2 / 4c_1$, taking explicit account of the constraint $u \geq 0$.

(BEEM103 January 2010) (*End of the exam paper.*)