test exam

OPTIMIZATION TECHNIQUES
FOR ECONOMISTS
Module convenor: Dieter Balkenborg

Duration: TWO HOURS

The paper has 3 parts. Your marks on the first part will be rounded down to 55 marks. Your marks on the second part will be rounded down to 15 marks. Obviously, you cannot get over 100 marks overall. This examination is closed book and no materials are allowed. Full work must be shown on your script. Please write legibly.
Part A (You can gain no more than 55 marks on this part.)

Problem 1 (10 marks) Simplify
\[ \frac{1}{8} \sqrt{x^2 y^{-1}} - \frac{2}{3} x^2 \sqrt[4]{y} \quad \text{and} \quad \frac{x/\sqrt[3]{x^2}}{x^5} \]

Problem 2 (10 marks) Solve
\[ x^3 - \ln x = x^2 + \ln x \]

Problem 3 (10 marks) Consider the function
\[ y(x) = xe^{-\frac{x^2}{2}} \]

i) Calculate and draw a sign diagram for the first derivative. Where is the function increasing or decreasing? Are there any peaks or troughs? Is the function quasi concave on the interval of positive numbers \( x > 0 \)? Does the function have a global maximum over the positive numbers \( x > 0 \)?

ii) Calculate and draw a sign diagram for the second derivative. Where is the function convex or concave? Are there any inflection points? Hint: The second derivative can be written in the form \( \frac{2q(x)}{(x^2+1)^2} \) where \( q(x) \) is a quadratic function.

Problem 4 (10 marks) For the function
\[ y = x^4 - 2x^2 \]
find the (global) maxima and minima a) on the interval \([-1, 1]\) and b) on the interval \([0, 2]\).

Problem 5 (10 marks) Find the equation of the tangent plane of
\[ z(x, y) = \ln(5x + y) + \ln(10x + y) \]
at the point \((x^*, y^*, z^*) = (2, 3, z(2, 3))\).

Problem 6 (10 marks) The only grocery store in a small rural community carries two brands of frozen apple juice, a local brand that it obtains at the cost of 30 cent per can and a well-known national brand that it obtains at a cost of 40 cent a can. The grocer estimates that if the local brand is sold for \( x \) cents per can and the national brand for \( y \) cents per can, approximately \((70 - 5x + 4y)\) cans of the local brand and \((80 + 6x - 7y)\) cans of the national brand will be sold each day. How should the grocer price each brand to maximize the profit from the sale of the juice?

(BEEM103 January 2010) (Please turn over.)
Problem 7 (10 marks) Find a solution to the differential equation
\[
\frac{dx}{dt} = t^4 x^2
\]

Problem 8 (10 marks) Solve the problem
\[
\min_{x(t)} \int_0^1 (1 + x + x^2 + \dot{x} + \dot{x}^2) e^{rt} dt
\]
subject to the restrictions \( x(0) = 0, x(1) = 1 \).

Part B (You can gain no more than 15 marks on this part.)

Problem 9 (15 marks) Determine the optimal profit for a firm with the production function
\[Q = A \ln K + B \ln L\]
\((A, B > 0)\) operating in perfectly competitive input- and output markets and facing the output price \( P > 0 \), the interest rate \( r > 0 \) and the wage rate \( w \).

Problem 10 (15 marks) For the point \((1, -3)\) find the point \((x^*, y^*)\) that minimizes the distance
\[
\sqrt{(x - a)^2 + (y - b)^2}
\]
subject to the constraints
\[
-1 \leq x \leq 1 \\
x - 1 \leq y \leq x + 1.
\]

Problem 11 (15 marks) Solve the problem
\[
\max_{u(t)} \int_0^1 -(x - 5)^2 dt
\]
subject to \( \dot{x} = u, \ x(0) = 0, \ x(1) = 2, \ -1 \leq u \leq 3 \).
Part C

Problem 12 (20 marks) Derive the demand function of a consumer with utility function

\[ u(x, y) = \sqrt{x} + \sqrt{y} \]

Problem 13 (20 marks) Solve the production planning problem

\[ \min \int_0^T \left( c_1 u^2 + c_2 x \right) dt \]

subject to \( x = u \), \( x(0) = 0 \), \( x(T) = B \), \( u(t) \geq 0 \) in the case \( B < c_2 T^2 / 4c_1 \), taking explicit account of the constraint \( u \geq 0 \).