BEEM103
UNIVERSITY OF EXETER

BUSINESS School
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Sample exam Solutions, Part A

OPTIMIZATION TECHNIQUES FOR ECONOMISTS

Part A (You can gain *no more than 55 marks* on this part.)

Problem 1 (10 marks) Simplify

$$\frac{8\sqrt[3]{x^2}\sqrt[4]{y}\sqrt{1/z}}{-2\sqrt[3]{x}\sqrt{y^5}\sqrt{z}} \qquad \left(\left((3a)^{-1}\right)^{-2}\left(2a^{-2}\right)^{-1}\right)/a^{-3}$$

Solution 1

$$\begin{aligned}\frac{8\sqrt[3]{x^2}\sqrt[4]{y}\sqrt{1/z}}{-2\sqrt[3]{x}\sqrt{y^5}\sqrt{z}} &= -4\frac{x^{\frac{2}{3}}}{x^{\frac{1}{3}}}\frac{y^{\frac{1}{4}}}{y^{\frac{5}{2}}}\frac{z^{-\frac{1}{2}}}{z^{\frac{1}{2}}} \\ &= -4x^{\frac{1}{3}}y^{-\frac{9}{4}}z^{-1} = -4\frac{\sqrt[3]{x}}{\sqrt[4]{y^9}z}\end{aligned}$$

$$\left(\left((3a)^{-1}\right)^{-2}\left(2a^{-2}\right)^{-1}\right)/a^{-3} = \left(9a^2 \times \frac{1}{2}a^2\right)a^3 = \frac{9}{2}a^7$$

Problem 2 (10 marks) Solve

$$\ln x^{5/2} - 0.5\ln x = \ln 25$$

Solution 2 We have

$$\ln x^{5/2} - 0.5\ln x = \frac{5}{2}\ln x - 0.5\ln x = 2\ln x$$

and $\ln 25 = \ln 5^2 = 2\ln 5$. Hence

$$2\ln x = 2\ln 5$$

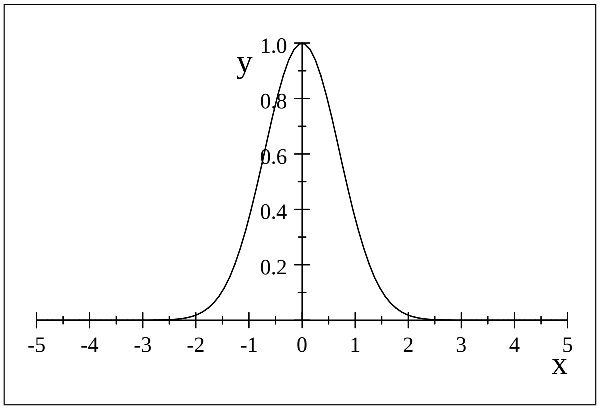
and so $x = 5$.

Problem 3 (10 marks) Consider the function

$$y(x) = e^{-x^2}$$

i) Calculate and draw a sign diagram for the first derivative. Where is the function increasing or decreasing? Are there any peaks or troughs? Does the function have a (global) maximum? Is the function quasi concave?
ii) Calculate and draw a sign diagram for the second derivative. Where is the function convex or concave? Are there any inflection points?

Solution 3



a) $y' = -2xe^{-x^2}$. Thus the function has a unique critical point at zero. From the sign of the derivative we see that the function is increasing to the left and decreasing to the right. Hence $x = 0$ is the (global) maximum of the function. Let $g(u) = e^u$ and $h(x) = -x^2$. $g(u)$ is increasing and $h(u)$ is concave. As a monotone transformation of a concave function the function $y(x) = e^{-x^2} = g(h(x))$ is quasi concave.

b) $y'' = 2(-1 + 2x^2)e^{-x^2}$. The second derivative is zero at $x = \pm\frac{1}{\sqrt{2}}$, in between it is negative, outside positive. $x = \pm\frac{1}{\sqrt{2}}$ are hence inflection points with the function concave in between the roots and convex outside.

Problem 4 (10 marks) For the function

$$y = \frac{1}{4}x^4 - 2x^2$$

find the (global) maxima and minima a) on the interval $[-1, 1]$ and b) on the interval $[-4, 4]$.

Solution 4

$$\frac{dy}{dx} = x^3 - 4x = x\left(x^2 - 4\right) = x(x+2)(x-2)$$

The function has critical points at $x = 0, +2, -2$. We have

$$\begin{aligned} y(\pm 4) &= 32 \\ y(\pm 2) &= -4 \\ y(0) &= 0 \end{aligned}$$

On the interval $[-2, 2]$ the minima are hence at ± 2 and the maximum is at $x = 0$. On the interval $[-4, 4]$ the minima are hence at ± 2 and the maxima are at $x = \pm 4$.

Problem 5 (10 marks) Find the equation of the tangent plane of

$$z(x, y) = \ln(5x + y)$$

at the point $(x^*, y^*, z^*) = (2, 3, z(2, 3))$.

Solution 5 $z^* = \ln 13$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{5}{5x+y}, \ \frac{\partial z}{\partial x}_{|x=2,y=3} = \frac{5}{5\times 2+3} = \frac{5}{13} \\ \frac{\partial z}{\partial y} &= \frac{1}{5x+y}, \ \frac{\partial z}{\partial x}_{|x=2,y=3} = \frac{1}{5\times 2+3} = \frac{1}{13}\end{aligned}$$

The formula for the total differential at $x = 2, y = 3$

$$dz = \frac{\partial z}{\partial x}_{|x=2,y=3} dx + \frac{\partial z}{\partial y}_{|x=2,y=3} dy$$

yields the formula for the tangent

$$\begin{aligned}(z - \ln 13) &= \frac{5}{13}(x-2) + \frac{1}{13}(y-3) \\ z &= \frac{5}{13}x + \frac{1}{13}y - 1 - \ln 13\end{aligned}$$

Problem 6 (10 marks) Show that the function

$$u(x,y) = \ln(5x+y) + \ln(x+y)$$

is concave.

Solution 6 We have

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{1}{x+y} + \frac{5}{5x+y} \\ \frac{\partial u}{\partial y} &= \frac{1}{x+y} + \frac{1}{5x+y} \\ \frac{\partial^2 u}{\partial x^2} &= -\frac{1}{(x+y)^2} - \frac{25}{(5x+y)^2} < 0 \\ \frac{\partial^2 u}{\partial x \partial y} &= -\frac{1}{(x+y)^2} - \frac{5}{(5x+y)^2} < 0 \\ \frac{\partial^2 u}{\partial y^2} &= -\frac{1}{(x+y)^2} - \frac{5}{(5x+y)^2} < 0\end{aligned}$$

Set $a = (x+y)^{-1}$ and $b = (5x+y)^{-1}$. The Hessian of $\hat{u}$ is

$$H = \begin{bmatrix} -a^2 - 25b^2 & -a^2 - 5b^2 \\ -a^2 - 5b^2 & -a^2 - b^2 \end{bmatrix}$$

and hence

$$\begin{aligned}\det H &= \left(a^2 + 25b^2\right)\left(a^2 + b^2\right) - \left(a^2 + 5b^2\right)^2 \\ &= a^4 + 26a^2b^2 + 25b^4 - a^4 - 10a^2b^2 - 25b^4 = (2a)^2 (2b)^2 > 0\end{aligned}$$

We see that the leading principle minors of the Hessian have the right sign and so $\hat{u}$ is concave, as was to be proved.

Problem 7 (10 marks) Find a solution to the differential equation

$$e^x \frac{dx}{dt} = t$$

Solution 7

$$\begin{aligned} \int e^x dx &= \int t dt \\ e^x &= \frac{1}{2}t^2 + \tilde{C} \\ x &= \ln \left| \frac{1}{2}t^2 + \tilde{C} \right| \end{aligned}$$

Problem 8 (10 marks) Solve the problem

$$\max \int_0^1 \left(1 - x^2 - \dot{x}^2\right) dt,\ x(0) = 0,\ x(1) \geq 0$$

Solution 8

$$\begin{aligned} F &= \left(1 - x^2 - \dot{x}^2\right) \\ F_x &= -2x \\ F_{\dot{x}} &= -2\dot{x} \\ \frac{d}{dt}F_{\dot{x}} &= -2\ddot{x} \end{aligned}$$

The Euler equation is

$$x = \ddot{x}$$

This is a homogeneous linear differential equation of second order with constant coefficients. The characteristic polynomial is $r^2 - 1 = (r-1)(r+1)$. The general solution to the Euler equation is hence

$$x(t) = Ae^r + Be^{-r}$$

To have $x(t) = 0$ we need $A = -B$. If $x(1) > 0$ we would need

$$F_{\dot{x}}(1) = -2\dot{x}(1) = -2\left(Ae - Be^{-1}\right) = 0$$

which would imply $Ae^2 = B = -A$ which can hold only if $A = 0$, contradicting $x(1) > 0$. We see that the only solution is $A = B = 0$ or $x(t) = 0$ for all t.

Part B (You can gain *no more than 15 marks* on this part.)

Problem 9 (15 marks) For a consumer with the utility function

$$u(x, y) = -(10 - x)^2 - (16 - 2y)^2$$

maximize his utility subject to the budget equation $b - p_x x - p_y y \leq b$ when a) $p_x = p_y = 25$ and $b = 100$; b) $p_x = 60$, $p_y = 25$ and $b = 100$ or c) $p_x = p_y = 1$ and $b = 100$.

Solution 9 In general the Lagrangian for this problem is

$$\mathcal{L} = -(10-x)^2 - (16-2y)^2 + \lambda_1 (b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y$$

which yields the first order conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 2(10-x) - \lambda_1 p_x + \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 8(8-y) - \lambda_1 p_y + \lambda_3 = 0 \end{aligned}$$

which must be satisfied together with the complementarity conditions

$$\begin{aligned} \lambda_1 (b - p_x x - p_y y) &= 0 \\ \lambda_2 x &= 0 \\ \lambda_3 y &= 0 \end{aligned}$$

Notice that $\mathcal{L}$ is a sum of concave functions and hence concave. The Kuhn-Tucker conditions are hence sufficient for an optimum.

Typically we expect only the budget constraint to be binding. In this case $\lambda_1 = \lambda_2 = 0$ and we obtain the first order conditions

$$\begin{aligned} 2(10-x) &= \lambda_1 p_x \\ 8(8-y) &= \lambda_1 p_y \end{aligned}$$

implying

$$\frac{2(10-x)}{8(8-y)} = \frac{p_x}{p_y} \Longleftrightarrow p_y(10-x) - 4p_x(8-y) = 0 \Longleftrightarrow 10p_y - 32p_x = p_y x - 4p_y y$$

which must be satisfied together with the budget equation

$$\begin{aligned} 10p_y - 32p_x &= p_y x - 4p_y y \\ p_x x + p_y y &= b \end{aligned}$$

a) $p_x = p_y = 25$ and $b = 100$. The above two equations yield

$$\begin{aligned} 10 \times 25 - 32 \times 25 &= 25x - 4 \times 25y \\ 25x + 25y &= 100 \end{aligned}$$

or

$$\begin{aligned} 10 - 32 &= x - 4y \\ x + y &= 4 \end{aligned}$$

Hence

$$
\begin{aligned}
x &= 4 - y \\
-22 &= 4 - y - 4y \\
26 &= 5y \\
y &= \frac{26}{5} \\
x &= \frac{20}{5} - \frac{26}{5} = -\frac{6}{5}
\end{aligned}
$$

The non-negativity constraint for x is violated, hence there is no solution where only the budget constraint binds.

Let us next assume that both the non-negativity constraint $x \geq 0$ and the budget constraint are binding. So $x = 0$ and the budget equation implies $25y = 100 \Longleftrightarrow y = 4$. Complementarity $\lambda_3 y = 0$ implies $\lambda_3 = 0$. From the second first order condition

$$8\,(8 - y) - 25\lambda_1 = 0$$

we get $\lambda_1 = 8\,(8-4)\,/25 = 32/25$. From the first first order condition

$$2\,(10 - 0) - 25 \times \frac{32}{25} + \lambda_2 = 0$$

we get $\lambda_2 = 32 - 20 = 12 > 0$.

Overall we get the solution $(x^*, y^*) = (0, 4)$, $(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{32}{25}, 12, 0\right) \geq 0$. With these numbers we see that all Kuhn-Tucker conditions are satisfied. We have found the optimum.

b) $p_x = 60$, $p_y = 25$ and $b = 100$. This is an even higher price on x than before. We assume $(x^*, y^*) = (0, 4)$ and $\lambda_3 = 0$. Then the budget equation and the non-negativity constraint for x are binding. It remains to check that all Lagrange multipliers are non-negative. The second order conditions yields again $\lambda_1 = 32/25$. The first second order condition yields

$$2\,(10 - 0) - 60 \times \frac{32}{25} + \lambda_2 = 0 \Longleftrightarrow \lambda_2 = 60 \times \frac{32}{25} - 20 \approx 56.8 > 0$$

We found the optimum.

c) $p_x = p_y = 1$ and $b = 100$. If we assume that only the budget constraint binds we get the two equations

$$
\begin{aligned}
10 - 32 &= x - 4y \\
x + y &= 100
\end{aligned}
$$

Thus

$$
\begin{aligned}
x &= 100 - y \\
22 &= 100 - 5y \\
5y &= 78 \\
y &= 78/5 \\
x &= 100 - 78/5
\end{aligned}
$$

which looks nice except that the second first order condtion yields with $\lambda_2 = \lambda_3 = 0$ that

$$\begin{aligned} 2(10-x) - \lambda_1 p_x + \lambda_2 &= 2(10 - 100 + 78/5) - \lambda_1 = 0 \\ &\iff \lambda_1 = 2(10 - 100 + 78/5) \approx -148.8 < 0 \end{aligned}$$

which is ruled out.

A negative shadow price on the budget constraint suggest that it should not be binding in optimum. Let us hence assume that no constraint is binding. Thus $\lambda_1 = \lambda_2 = \lambda_3 = 0$. The first order conditions yield $x^* = 10$, $y^* = 8$. Both quantities are strictly positive and also satisfy the budget equation:

$$1 \times 10 + 1 \times 8 < 100$$

Hence $(x^*, y^*) = (10, 8)$ is the optimum.

Problem 10 (15 marks) Solve the problem

$$\min \int_0^7 (x(t) - 3)^2 \text{ subject to } \dot{x}(t) = u(t),\ x(0) = x(7) = 0,\ -1 \leq u(t) \leq 1$$

Solution 10 The Hamiltonian is (because we want to maximize the negative of the integral)

$$H = -(x-3)^2 + qu$$

Notice that the Hamiltonian is concave in x and u. (Why?) We obtain because H is linear in u

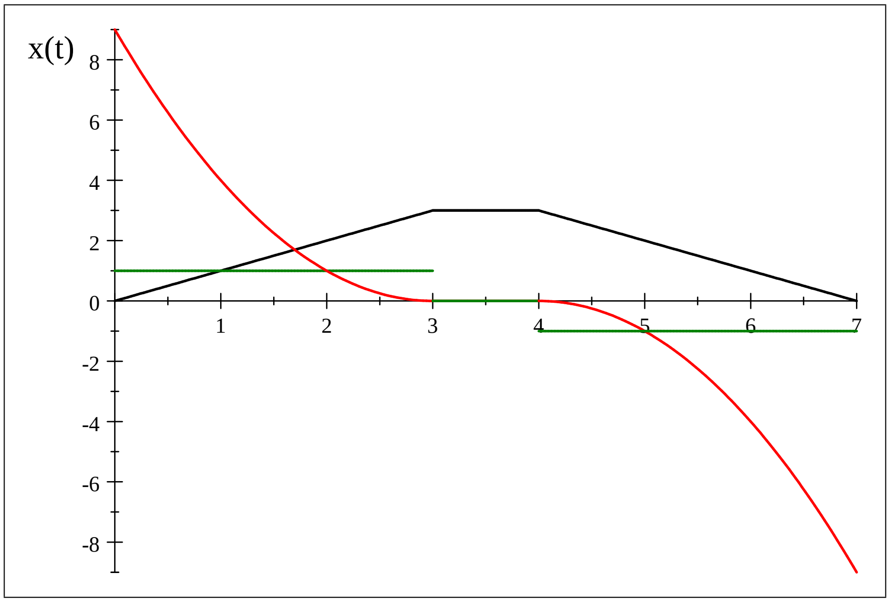
$$\arg \max_{-1 \leq u \leq 1} H = \begin{cases} 1 & \text{for } q > 0 \\ \in [-1, 1] & \text{for } q = 0 \\ -1 & \text{for } q < 0 \end{cases}$$

Moreover, we need

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial q} = u \\ \dot{q} &= -\frac{\partial H}{\partial x} = -2(x-3) \end{aligned}$$

It is natural to assume that $x(t)$ first increases as fast as possible (i.e., at rate $\dot{x} = 1$) until it it reaches the value $x = 3$ where $-(x-3)^2$ is minimized, stays at this value as long as possible at this value and then decreases at the maximal rate $\dot{x} = 1$ until $x(t) = 0$ at $t = 7$. This suggest the solution

$$x(t) = \begin{cases} t & \text{for } 0 \leq t \leq 3 \\ 3 & \text{for } 3 \leq t \leq 4 \\ 3 - t & \text{for } 4 \leq t \leq 7 \end{cases} \qquad u(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 3 \\ 0 & \text{for } 3 \leq t \leq 4 \\ -1 & \text{for } 4 \leq t \leq 7 \end{cases}$$



The optimal functions $x(t)$ (black), $u(t)$ (green) and $q(t)$ (red)

To verify the above conditions we want to find a corresponding function $q(t)$ such that $\dot{q} = -\frac{\partial H}{\partial x}$ holds. Because u must maximize the Hamiltonian we must have

$$q(t) \begin{cases} > 0 & \text{for} \quad 0 \le t \le 3 \\ 0 & \text{for} \quad 3 \le t \le 4 \\ < 0 & \text{for} \quad 4 \le t \le 7 \end{cases}$$

For $t \le 3$ we have

$$\begin{aligned} \dot{q} &= -\frac{\partial H}{\partial x} = -(-2(x(t) - 3)) = 2(t-3) \\ q(t) &= 2\int (t-3)\,dt = t(t-6) + C \end{aligned}$$

We must have $q(3) = 0$ and so $C = 3 \times 3 = 9$, which yields $q(t) = t^2 - 6t + 9 = (t-3)^2$. For $3 \le t \le 4$ we have

$$\dot{q} = -\frac{\partial H}{\partial x} = 0$$

To have $q(3) = 0$ we get $q(t) = 0$ in this range. For $t \ge 4$

$$\begin{aligned} \dot{q} &= -\frac{\partial H}{\partial x} = -(-2(x(t) - 3)) = 2(7 - t - 3) = 2(4-t) \\ q(t) &= 2\int (4-t)\,dt = t(8-t) + C \end{aligned}$$

and since $q(4) = 4 \times 4 + C = 0$ we get $q(t) = -t^2 + 8t - 16 = -(t-4)^2$. Overall,

$$q(t) = \begin{cases} (t-3)^2 & \text{for} \quad 0 \le t \le 3 \\ 0 & \text{for} \quad 3 \le t \le 4 \\ -(t-4)^2 & \text{for} \quad 4 \le t \le 7 \end{cases}$$

We see that all the conditions required by the optimal control approach are satisfied: $u(t)$ maximizes the Hamiltonian for all t, $\dot{x} = \frac{\partial H}{\partial q}$, $\dot{q} = -\frac{\partial H}{\partial x}$ and $x(0) = x(1) = 0$. Because the Hamiltonian has the required concavity requirements we conclude that we have found the optimal solutions.

Part C

Problem 11 (20 marks) Sketch the graph of the area C carved out by the two inequalities

$$\begin{aligned} x^2 + (y-1)^2 &\leq 4 \\ x^2 + (y+1)^2 &\leq 4 \end{aligned}$$

For any point (a, b) in the plane use the Lagrangian approach to determine the point closest to (a, b) within or on the boundary of C. Why can you assume without loss of generality that $a, b \geq 0$? How many cases do we have to consider? Give an argument why you can assume without loss of generality that $a, b \geq 0$.

Solution 11 Four cases must be considered: No constraint is binding, one of the two is binding, or both are binding. By symmetry it is sufficient to do the calculations for $(a, b) \geq 0$. Instead of minimizing the distance we can maximize the negative of the square of the distance and so the Lagrangian is

$$\mathcal{L} = -(x-a)^2 - (y-b)^2 + \lambda_1 \left(4 - x^2 - (y-1)^2\right) + \lambda_2 \left(4 - x^2 - (y+1)^2\right)$$

the FOC are

$$\begin{aligned} -2(x-a) - 2\lambda_1 x - 2\lambda_2 x &= 0 \\ -2(y-b) - 2\lambda_1 (y-1) - 2\lambda_2 (y+1) &= 0 \end{aligned}$$

If no constraint is binding we set $\lambda_1 = \lambda_2 = 0$ and obtain $x^* = a$, $y^* = b$. This is the solution if (a, b) is in the area C. For ($a \geq 0$ and $b > 0$) only the constraint $x^2 + (y+1)^2 - 4 \geq 0$ can be binding, which is obvious if one sketches C. The FOC become, after setting $\lambda_1 = 0$

$$\begin{aligned} -2(x-a) - 2\lambda_2 x &= 0 \\ -2(y-b) - 2\lambda_2 (y+1) &= 0 \end{aligned}$$

Thus

$$\begin{aligned} \lambda_2 x &= -(x-a) \\ \lambda_2 &= -\frac{x-a}{x} \\ \lambda_2 &= -\frac{y-b}{y+1} \\ \frac{x-a}{x} &= \frac{y-b}{y+1} \\ (x-a)(y+1) &= x(y-b) \\ -ay + x - a &= -bx \\ (1+b)x &= (1+y)a \\ 1+y &= (1+b)\frac{x}{a} \end{aligned}$$

$$
\begin{aligned}
x^2 + (1+y)^2 &= 4 \\
a^2x^2 + (1+b)^2 x^2 &= 4a^2 \\
\left[a^2 + (1+b)^2\right] x^2 &= 4a^2 \\
x &= \frac{2a}{\sqrt{a^2 + (1+b)^2}} \\
y + 1 &= \frac{2(1+b)}{\sqrt{a^2 + (1+b)^2}}
\end{aligned}
$$

For this to be a solution we need

$$
\begin{aligned}
\frac{2(1+b)}{\sqrt{a^2 + (1+b)^2}} - 1 &\geq 0 \\
2(1+b) &\geq \sqrt{a^2 + (1+b)^2} \\
4(1+b)^2 &\geq a^2 + (1+b)^2 \\
3(1+b)^2 &\geq a^2 \\
\sqrt{3}(1+b) &\geq a
\end{aligned}
$$

One can check that for $-\sqrt{3}(1+b) \leq a \leq \sqrt{3}(1+b)$ both constraints are binding and so the solution is $x^* = \sqrt{3}$, $y^* = 0$, where the two circles meet.

Problem 12 (20 marks) Two factors, capital, $K(t)$, and an extractive resource, $R(t)$, are used to produce a good, Q, according to the production function $AK^{1-\alpha}R^{a}$ where $0 < \alpha < 1$. The product may be consumed, yielding utility $U(C) = \ln C$, or it may be invested as capital. The total amount of the extractive resource is X_0. Maximize over the finite horizon T utility

$$
\int_0^T \ln C(t)\, dt
$$

subject to $X' = -R$, $X(0) = X_0$, $X(T) = 0$, $K' = AK^{1-\alpha}R^{\alpha} - C$, $K(0) = K_0$, $C > 0$, $R > 0$. (All parameters are assumed to be positive.)

Solution 12 See Kamian Schwartz p. 138. for more details.,

The Hamiltonian is, after substituting $y = R/K$

$$
H = \ln C - \lambda_1 K y + \lambda_2 (AKy^{\alpha} - C)
$$

which gives the FOC

$$
\begin{aligned}
0 &= \frac{\partial H}{\partial C} = \frac{1}{C} - \lambda_2 & (1) \\
0 &= \frac{\partial H}{\partial y} = -\lambda_1 K + \lambda_2 \alpha A K y^{\alpha-1} & (2) \\
\frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial X} = 0 & (3) \\
\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial K} = \lambda_1 y - \lambda_2 A y^{\alpha} & (4)
\end{aligned}
$$

(3) yields that λ_1 is constant. If $K \neq 0$ then (2) yields

$$\begin{aligned}
\lambda_1 &= \lambda_2 \alpha A y^{\alpha-1} \\
0 &= \frac{d\lambda_1}{dt} = \frac{d\lambda_2}{dt}\alpha A y^{\alpha-1} + \lambda_2 \alpha A (\alpha-1) y^{\alpha-2}\frac{dy}{dt} \\
\frac{d\lambda_2}{dt}/\lambda_2 &= -\frac{(\alpha-1) y^{-\alpha-2}}{y^{\alpha-1}}\frac{dy}{dt} = (1-\alpha)\frac{dy}{dt}/y
\end{aligned}$$

Substituting the first of the above equations into (4) we also get

$$\begin{aligned}
\frac{d\lambda_2}{dt} &= \lambda_2 \alpha A y^{\alpha-1} y - \lambda_2 A y^{\alpha} = -(1-\alpha) A y^{\alpha} \\
\frac{d\lambda_2}{dt}/\lambda_2 &= -(1-\alpha) A y^{\alpha}
\end{aligned}$$

Thus, combining the last two results

$$\begin{aligned}
\frac{d\lambda_2}{dt}/\lambda_2 &= (1-\alpha)\frac{dy}{dt}/y = -(1-\alpha) A y^{\alpha} \\
\frac{dy}{dt} &= -A y^{\alpha+1} \\
\frac{dy}{y^{\alpha+1}} &= -A dt \\
\int \frac{1}{y^{\alpha+1}} dy &= -\int A dt \\
-\frac{1}{\alpha} y^{-\alpha} &= -At + k_1 \\
y^{\alpha} &= \frac{1}{\alpha A t + \alpha k_1}
\end{aligned}$$

$$\begin{aligned}
\frac{d\lambda_2}{dt}/\lambda_2 &= -(1-\alpha) A y^{\alpha} = -\frac{1-\alpha}{\alpha t + \alpha k_1/A} \\
\ln \lambda_2 &= -(1-\alpha)\int \frac{1}{\alpha t + \alpha k_1/A} dt = -\frac{1-\alpha}{\alpha}\ln(\alpha t + \alpha k_1/A) + k_2 \\
\lambda_2 &= k_2 (\alpha t + \alpha k_1/A)^{-(1-\alpha)/\alpha}
\end{aligned}$$

and so

$$\begin{aligned}
C &= 1/\lambda_2 = k_2 (\alpha t + \alpha k_1/A)^{(1-\alpha)/\alpha} \\
K' &= A K y^{\alpha} - C \\
K' &= A K \frac{1}{\alpha A t + \alpha k_1} - k_2 (\alpha t + \alpha k_1/A)^{(1-\alpha)/\alpha} \\
\frac{dK}{dt} &= \gamma \frac{K}{t+d} - \delta (t+d)^{(1-\alpha)/\alpha}
\end{aligned}$$

Exact solution is:

$$K(t) = -\frac{(t+d)^{\gamma - \frac{-1+\alpha+\gamma\alpha}{\alpha}+1}}{\frac{1}{\alpha}-\gamma}\delta + (t+d)^{\gamma} C_1$$