BEEM103
UNIVERSITY OF EXETER
BUSINESS SCHOOL
December 2009

OPTIMIZATION TECHNIQUES FOR ECONOMISTS

Mock Exam - Solutions (except questions B2, B3)
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Part A (You can gain *no more than 55 marks* on this part.)

Problem 1 (10 marks) Simplify

$$\frac{x\sqrt{y}-y\sqrt{x}}{x\sqrt{y}+y\sqrt{x}} \qquad \frac{\sqrt[3]{a}\left((4a)^6\right)^{\frac{1}{12}}\sqrt[4]{a^3}}{(4a^5)^{\frac{1}{12}}\sqrt{a}}$$

Solution 1

$$\begin{aligned}
\frac{x\sqrt{y}-y\sqrt{x}}{x\sqrt{y}+y\sqrt{x}} &= \frac{\left(x\sqrt{y}-y\sqrt{x}\right)\left(x\sqrt{y}-y\sqrt{x}\right)}{\left(x\sqrt{y}+y\sqrt{x}\right)\left(x\sqrt{y}-y\sqrt{x}\right)} = \\
\frac{\left(x\sqrt{y}\right)^2-2\left(x\sqrt{y}\right)\left(y\sqrt{x}\right)+\left(y\sqrt{x}\right)^2}{\left(x\sqrt{y}\right)^2-\left(y\sqrt{x}\right)^2} &= \frac{x^2y-2xy\sqrt{x}\sqrt{y}+xy^2}{x^2y-xy^2} = \\
\frac{y-2\sqrt{xy}+x}{x-y} &= \frac{\left(\sqrt{x}-\sqrt{y}\right)^2}{x-y}
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt[3]{a}\left((4a)^6\right)^{\frac{1}{12}}\sqrt[4]{a^3}}{(4a^5)^{\frac{1}{12}}\sqrt{a}} &= \frac{a^{\frac{1}{3}}\left(4^6a^6\right)^{\frac{1}{12}}a^{\frac{3}{4}}}{(4a^5)^{\frac{1}{12}}a^{\frac{1}{2}}} = \frac{4^{\frac{6}{12}}a^{\frac{1}{3}}a^{\frac{6}{12}}a^{\frac{3}{4}}}{4^{\frac{1}{12}}a^{\frac{5}{12}}a^{\frac{1}{2}}} = \\
4^{\frac{6}{12}-\frac{1}{12}}a^{\frac{1}{3}+\frac{6}{12}+\frac{3}{4}-\frac{5}{12}-\frac{1}{2}} &= \left(2^2\right)^{\frac{5}{12}}a^{\frac{4+1+9-5-6}{12}} = 2^{\frac{5}{6}}a^{\frac{3}{12}} = \sqrt[6]{2^5}\sqrt[4]{a}
\end{aligned}$$

Problem 2 (10 marks) Solve

$$\begin{aligned}
\ln x^2 - \ln y^2 &= \ln 25 \\
x - y &= 4
\end{aligned}$$

Solution 2

$$\begin{aligned}
x^2/y^2 &= 25 \\
x^2 &= 25y^2 \\
(y+4)^2 &= 25y^2 \\
24y^2-8y-16 &= 0 \\
3y^2-y-2 &= 0 \\
y &= \frac{+1\pm\sqrt{(-1)^2+4\times3\times2}}{2\times3} = \frac{+1\pm\sqrt{25}}{6} = \frac{+1\pm5}{6}
\end{aligned}$$

Thus $y=\frac{6}{6}=1$ because the solution must be positive since $\ln y^2$ is in one of the equations and $y=\frac{1-5}{6}$ is therefore ruled out. Moreover, $x=4+y=5$.

Problem 3 (10 marks) Consider the function

$$y(x) = \frac{1}{1+x^2}$$

i) Calculate and draw a sign diagram for the first derivative. Where is the function increasing or decreasing? Are there any peaks or troughs? Does the function have a (global) maximum. Is the function quasi concave?

ii) Calculate and draw a sign diagram for the second derivative. Where is the function convex or concave. Are there any inflection points?

Solution 3 We have

$$y'(x) = -\left(1+x^2\right)^{-2} \times 2x = -\frac{2x}{(1+x^2)^2}$$

We see that the function has a positive derivative and is hence decreasing until $x = 0$. Thereafter the derivative is negative and the function decreasing. In particular, the function is quasi-concave with a global maximum at zero. (Alternative proof: The function $h(z) = \frac{1}{1-z}$ is strictly increasing for all numbers $z < 1$ (a convex set or interval!) because $h'(z) = (1-z)^{-2} > 0$. The function $-x^2$ is concave and its monotone transformation $h(-x^2) = \frac{1}{1+x^2}$ therefore quasi-concave.

The second derivative is

$$y''(x) = -\frac{2(1+x^2)^2 - 2x \times 2(1+x^2)(2x)}{(1+x^2)^4} = -2\frac{(1+x^2) - 4x^2}{(1+x^2)^3} = 2\frac{3x^2-1}{(1+x^2)^3}$$

We see that there are inflection points at $x = \pm\frac{1}{\sqrt{3}}$. For $x > \frac{1}{\sqrt{3}}$ or $x < -\frac{1}{\sqrt{3}}$ the function is convex. For $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$ the function is concave.

Problem 4 (10 marks) For the function

$$y(x) = \frac{1}{3}x^3 - x$$

find the (global) maxima and minima a) on the interval $[-2, 0]$ and b) on the interval $[-2, 2]$.

Solution 4

$$y'(x) = x^2 - 1 = (x+1)(x-1)$$

The function has critical points at $x = 1$ and at $x = -1$. We have $y(-2) = -\frac{8}{3} + 2 = -\frac{2}{3}$, $y(0) = 0$, $y(-1) = +\frac{2}{3}$. On this interval the global maximum is at $x = -1$ and the global minimum at $x = -2$. We have $y(-2) = -\frac{2}{3}$, $y(2) = \frac{2}{3}$, $y(-1) = +\frac{2}{3}$, $y(1) = -\frac{2}{3}$. On this interval a global maximum is at $x = 2$ and $x = -1$ and a global minimum at $x = -2$ and $x = 1$.

Problem 5 (5 marks) Find the equation of the tangent plane of

$$z(x, y) = x^2 y^2$$

at the point $(x^*, y^*, z^*) = (1, 4, z(1, 4))$.

Solution 5 $z(1,4) = 16$

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2xy^2, \ \frac{\partial z}{\partial x}_{|x=1,y=4} = 32 \\ \frac{\partial z}{\partial y} &= 2x^2y, \ \frac{\partial z}{\partial y}_{|x=1,y=4} = 8\end{aligned}$$

$$\begin{aligned}dz &= \frac{\partial z}{\partial x}_{|x=1,y=4} dx + \frac{\partial z}{\partial y}_{|x=1,y=4} dy \\ (z-16) &= 32(x-1) + 8(y-4) \\ z &= 32x + 8y - 48\end{aligned}$$

Problem 6 (10 marks) Show for the Cobb-Douglas production function

$$Q = K^{\frac{1}{3}} L^{\frac{1}{4}}$$

that the profit function

$$\Pi(K,L) = PK^{\frac{1}{3}} L^{\frac{1}{4}} - rK - wL$$

of price-taking firm facing the output price $P = 10$, the interest rate $r = 5$ and the wage rate $w = 2$ has a unique maximum.

Solution 6 For the concavity of the production function and hence the profit function (why?) see earlier material.

The profit function is

$$\Pi(K,L) = 10K^{\frac{1}{3}} L^{\frac{1}{4}} - 5K - 2L$$

The first order condition for a maximum are

$$\begin{aligned}\frac{\partial \Pi}{\partial K} &= \frac{10}{3} K^{-\frac{2}{3}} L^{\frac{1}{4}} - 5 = 0 \\ \frac{\partial \Pi}{\partial K} &= \frac{10}{4} K^{\frac{1}{3}} L^{-\frac{3}{4}} - 2 = 0\end{aligned}$$

Putting the prices on the right hand side and dividing we get

$$\begin{aligned}\frac{4}{3}\frac{L}{K} &= \frac{5}{2} \\ L &= \frac{15}{8} K\end{aligned}$$

Substituting this in the first order condition yields

$$\begin{aligned}
\frac{10}{3}K^{-\frac{2}{3}}\left(\frac{15}{8}\right)^{\frac{1}{4}}K^{\frac{1}{4}} &= 5 \\
K^{\frac{3-8}{12}} &= K^{-\frac{5}{12}} = 5\times\frac{3}{10}\times\left(\frac{8}{15}\right)^{\frac{1}{4}} \\
K &= \left(5\times\frac{3}{10}\times\left(\frac{8}{15}\right)^{\frac{1}{4}}\right)^{-\frac{12}{5}} \approx 0.551\,04 \\
L &= \frac{15}{8}\left(5\times\frac{3}{10}\times\left(\frac{8}{15}\right)^{\frac{1}{4}}\right)^{-\frac{12}{5}} \approx 1.033\,2
\end{aligned}$$

Because the function is concave this is the global maximum.

Problem 7 (10 marks) Find a solution to the differential equation

$$\frac{dx}{dt} = t^2x^2$$

with $x(1) = 1/2$.

Solution 7

$$\begin{aligned}
x^{-2}dx &= t^2dt \\
\int x^{-2}dx &= \int t^2dt \\
-x^{-1} &= \frac{1}{3}t^3 + c \\
x &= \frac{3}{C-t^3} \text{ where } C = \frac{c}{3}
\end{aligned}$$

Problem 8 (10 marks) Solve the problem

$$\max\int_0^1\left(10-\dot{x}^2-2x\dot{x}-5x^2\right)e^{-t}dt,\ x(0)=1,\ x(1)=1$$

Solution 8 $F = (10-\dot{x}^2-2x\dot{x}-5x^2)\,e^{-t}$

$$\begin{aligned}
\frac{dF}{dx} &= (-2\dot{x}-10x)\,e^{-t},\ \frac{\partial F}{\partial\dot{x}} = (-2\dot{x}-2x)\,e^{-t} \\
\frac{d}{dt}\left(\frac{\partial F}{\partial\dot{x}}\right) &= (-2\ddot{x}-2\dot{x})\,e^{-t} - (-2\dot{x}-2x)\,e^{-t}
\end{aligned}$$

Euler equation:

$$\begin{aligned}
(-2\dot{x}-10x)\,e^{-t} &= (-2\ddot{x}-2\dot{x})\,e^{-t} - (-2\dot{x}-2x)\,e^{-t} \\
0 &= -2\ddot{x}+2\dot{x}+12x \\
0 &= \ddot{x}-\dot{x}-6x
\end{aligned}$$

This linear differential equation is homogeneous and has constant coefficients. It characteristic equation is

$$0 = \alpha^2 - \alpha - 6 = (a-3)(a+2)$$

The solutions to the Euler equation are hence of the form

$$x(t) = Ae^{3t} + Be^{-2t}$$

The boundary condition $x(0) = 0$ yields $A = -B$ and from $x(1) = 1$ we obtain $1 = A(e^3 - e^{-2})t$. We obtain the unique candidate for a solution

$$x(t) = \frac{1}{e^3 - e^{-2}}\left(e^{3t} - e^{-2t}\right)$$

Part B (You can gain *no more than 15 marks* on this part.)

Problem 9 (15 marks) Consider the area defined by the inequalities

$$\begin{aligned} -1 &\leq x + y \leq 1 \\ -1 &\leq x \leq 1 \end{aligned}$$

Which point in this area is closest to $(5, -5)$?

Solution 9 I strongly recommend to draw a graph and get an idea which point in the area is closest!

The Lagrangian is

$$\mathcal{L} = -(x-5)^2 - (y+5)^2 + \lambda_1(1 - x + y) + \lambda_2(x + y + 1) + \lambda_3(1 - x) + \lambda_4(x + 1)$$

A graph gives me the conjecture that the closest point in the area is the point $(1, -2)$ where the two constraints $x = 1$ and $x + y = -1$ are binding and hence the complementarity conditions $\lambda_2(x + y + 1) = \lambda_3(1 - x) = 0$ hold. Notice that $(x, y) = (1, -2)$ is "feasible", i.e. satisfies all the above inequalities. For $(x, y) = (1, -2)$ we have $x + y < 0$ and $x > -1$, so $\lambda_1 = \lambda_3 = 0$ by the remaining complementarity conditions. The Lagrangian becomes

$$\mathcal{L} = -(x-5)^2 - (y+5)^2 + \lambda_2(x + y + 1) + \lambda_3(1 - x)$$

The first order conditions for the constrained maximum are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= -2(x-5) + \lambda_2 - \lambda_3 = -2 \times -4 + \lambda_2 - \lambda_3 = 8 + \lambda_2 - \lambda_3 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= -2(y+5) + \lambda_2 = -2 \times 3 + \lambda_2 = 0 \end{aligned}$$

Thus $\lambda_2 = 6 > 0$ from the last equation and the one above gives $\lambda_3 = 8 - 6 > 0$. In summary, all first order conditions and all complementarity conditions are satisfied. The solution is feasible and the Lagrange multipliers are non-negative. Since the Lagrangian is concave (why?) $(x, y) = (1, -2)$ is a constrained optimum.

Part C

Problem 10 (20 marks) Sketch the graph of the area C carved out by the two inequalities

$$\begin{aligned} x^2 + y^2 &\leq 4 \\ y &\leq x \end{aligned}$$

For any point (a, b) with $a, b \geq 0$ in the plane use the Lagrangian approach to determine the point closest to (a, b) within or on the boundary of C.

Solution 10 The Lagrangian is

$$\mathcal{L} = -(x-a)^2 - (y-b)^2 + \lambda_1 (x - y) + \lambda_2 \left(4 - x^2 - y^2\right)$$

The first order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= -2(x-a) + \lambda_1 - 2\lambda_2 x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= -2(y-b) - \lambda_1 - 2\lambda_2 y = 0 \end{aligned}$$

The complementarity conditions are

$$\begin{aligned} \lambda_1 (x-y) &= 0 \\ \lambda_2 \left(4 - x^2 - y^2\right) &= 0 \end{aligned}$$

CASE 1: No constraint binds. So $\lambda_1 = \lambda_2 = 0$. The first order conditions yield $a = x$, $b = y$. The point (a, b) must be in the area C.

CASE 2: Only the constraint $y \leq x$ binds. So $\lambda_2 = 0$. The FOC yield together with $x = y$

$$\begin{aligned} x - a &= -\lambda_1/2 = b - y = b - x \\ 2x &= b + a \\ x &= y = \frac{b+a}{2} \end{aligned}$$

(x, y) must satisfy $x^2 + y^2 \leq 4$, so $2\left(\frac{b-a}{2}\right)^2 = \frac{(b+a)^2}{2} \leq 4$ or $(b+a) \leq 8$. We must have, moreover,

$$\lambda_1 = 2(x-a) = 2\frac{b-a}{2} = b - a$$

So we must have $b \geq a$.

CASE 3: Only the constraint $x^2 + y^2 \leq 4$ binds. Thus $\lambda_1 = 0$. The FOC yield

$$\begin{aligned} a &= (1 + \lambda_2)\, x \\ b &= (1 + \lambda_2)\, y \end{aligned}$$

Division yields $\frac{x}{y} = \frac{a}{b}$ or $\frac{b}{a} = \frac{y}{x}$. (The only case missing here is $x = y = a = b = 0$, which was handled in CASE 1.) Moreover, for $b \neq 0$ in the first case

$$\begin{aligned}
4 &= x^2 + y^2 = x^2 + \left(\frac{b}{a}\right)^2 x^2 \\
x^2 &= \frac{4}{1 + \left(\frac{b}{a}\right)^2} = \frac{4a^2}{a^2 + b^2} \\
x &= \pm \frac{2a}{\sqrt{a^2 + b^2}} \\
y &= \pm \frac{2b}{\sqrt{a^2 + b^2}}
\end{aligned}$$

Because the Lagrange multiplier must be positive the FOC yield

$$x = \frac{2a}{\sqrt{a^2 + b^2}}, \, y = \frac{2b}{\sqrt{a^2 + b^2}}$$

To have $y \leq x$ this is a solution only if $b \leq a$.

Case 4: Both constraints bind. So $x = y$, $2x^2 = 4$, $x = y = \pm\sqrt{2}$. The FOC yield for $x = y = \sqrt{2}$

$$\begin{aligned}
-2\left(\sqrt{2} - a\right) + \lambda_1 - 2\sqrt{2}\lambda_2 &= 0 \\
-2\left(\sqrt{2} - b\right) - \lambda_1 - 2\sqrt{2}\lambda_2 &= 0
\end{aligned}$$

Addition gives

$$-4\sqrt{2} + 2\,(a + b) - 4\sqrt{2}\lambda_2 = 0$$
$$2\sqrt{2}\lambda_2 = -2\sqrt{2} + (a + b)$$

which implies $a + b \geq 2\sqrt{2}$ because λ_2 cannot be negative. We obtain from substitution

$$\begin{aligned}
-2\left(\sqrt{2} - a\right) + \lambda_1 + 2\sqrt{2} - (a + b) &= 0 \\
\lambda_1 + a - b &= 0
\end{aligned}$$

and because λ_1 must be non-negative, $a \geq b$. Similarly we obtain for the case $x = y = -\sqrt{2}$ that $a + b \leq -2\sqrt{2}$ and $a \geq b$.