The paper has 3 parts. Your marks on the first part will be rounded down to 55 marks. Your marks on the second part will be rounded down to 15 marks. Obviously, you cannot get over 100 marks overall. This examination is closed book and no materials are allowed. **Full work must be shown** on your script. **Please write legibly.**

Submit your answers in the first lecture in December 12th, Room 004, in the Harrison Building.
Part A (You can gain no more than 55 marks on this part.)

Problem 1 (5 marks) Simplify

\[
x \sqrt{y} - y \sqrt{x} - \frac{\sqrt[3]{a} \left((4a)^{\frac{1}{4}}\right)^3 \sqrt[3]{a^3}}{(4a^5)^{\frac{1}{4}} \sqrt[3]{a}}
\]

Problem 2 (5 marks) Solve

\[
\ln x^2 - \ln y^2 = \ln 25 \\
x - y = 4
\]

Problem 3 (10 marks) Consider the function

\[
y(x) = \frac{1}{1 + x^2}
\]

i) Calculate and draw a sign diagram for the first derivative. Where is the function increasing or decreasing? Are there any peaks or troughs? Does the function have a (global) maximum. Is the function quasi concave?

ii) Calculate and draw a sign diagram for the second derivative. Where is the function convex or concave. Are there any inflection points?

Problem 4 (10 marks) For the function

\[
y(x) = \frac{1}{3}x^3 - x
\]

find the (global) maxima and minima a) on the interval \([-2, 0]\) and b) on the interval \([-2, 2]\).

Problem 5 (5 marks) Find the equation of the tangent plane of

\[
z(x, y) = x^2 y^2
\]
at the point \((x^*, y^*, z^*) = (1, 4, z(1, 4))\).

Problem 6 (15 marks) Show for the Cobb-Douglas production function

\[
Q = K^{\frac{1}{4}} L^{\frac{3}{4}}
\]

that the profit function

\[
\Pi(K, L) = PK^{\frac{1}{4}} L^{\frac{3}{4}} - rK - wL
\]
of price-taking firm facing the output price \(P = 10\), the interest rate \(r = 5\) and the wage rate \(w = 2\) has a unique maximum.

(BEEM103 December 2009) (Please turn over.)
Problem 7 (10 marks) Find a solution to the differential equation
\[ \frac{dx}{dt} = t^2x^2 \]
with \( x(1) = 1/2 \).

Problem 8 (10 marks) Solve the problem
\[ \max \int_0^1 (10 - \ddot{x}^2 - 2\dot{x}\ddot{x} - 5x^2) e^{-t} dt, \ x(0) = 1, x(1) = 1 \]

Part B (You can gain no more than 15 marks on this part.)

Problem 9 (15 marks) Consider the area defined by the inequalities
\[
-1 \leq x + y \leq 1 \\
-1 \leq x \leq 1
\]
Which point in this area is closest to (5, -5)?

Problem 10 (15 marks) Find the only possible solution to
\[ \max \int (x^2 - 2u) dt \] subject to \( \dot{x} = u, \ x(0) = 0, \ x(1) \) free
Hint: Show that the co-state variable \( q(t) \) is strictly decreasing.

Problem 11 (15 marks) Use the Bordered Hessian to show that the function
\[ f(x, y) = e^{-\frac{1}{x+y^2}} \]
is quasiconcave.

Part C

Problem 12 (20 marks) Sketch the graph of the area \( C \) carved out by the two inequalities
\[
x^2 + y^2 \leq 4 \\
y \leq x
\]
For any point \((a, b)\) with \(a, b \geq 0\) in the plane use the Lagrangian approach to determine the point closest to \((a, b)\) within or on the boundary of \(C\).

(BEEM103 December 2009) (End of the exam paper.)