Exercise 1 Find the equations \( z = ax + c \) for the tangents to the graph of the function \( z = f(x) = x^3 - 4x \)

for \( x_0 = 1 \) and for \( x_0 = 3 \).

Solution: see solutions to week 3

Exercise 2 Find the equations \( z = ax + by + c \) for the tangent planes to the graph of the function

\( z = f(x, y) = x^3y - 4xy^2 \)

for \( (x_0, y_0) = (1, 2) \) and for \( (x_0, y_0) = (2, 1) \).

Solution: see solutions to week 3.

Exercise 3 For the function

\( z = xy \)

find the slope of the level curves in the points \((1, 2)\) and \((2, 2)\). Find the equations for the tangents.

Solution: see solutions to week 3.

Exercise 4 Suppose a consumer has the utility function \( u(x, y) = xy \) and his budget is 40. Given your knowledge of economics and the results from the previous question, what will be his demand when the prices are a) \( p_x = p_y = 10 \) and b) \( p_x = 40, p_y = 10 \)?

Solution: see solutions to week 3.

Exercise 5 Show that the function

\( Q = K^\alpha L^\beta \)

with \( \alpha, \beta > 0 \) and \( \alpha + \beta < 1 \) is concave.

Solution 1 It is assumed here that \( K \) and \( L \) are positive.

\[
\begin{align*}
\frac{\partial Q}{\partial K} &= \alpha K^{\alpha-1}L^\beta, \quad \frac{\partial Q}{\partial L} = \beta K^\alpha L^{\beta-1} \\
H &= \begin{bmatrix}
\frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K \partial L} \\
\frac{\partial^2 Q}{\partial L \partial K} & \frac{\partial^2 Q}{\partial L^2}
\end{bmatrix} = \begin{bmatrix}
\alpha (\alpha - 1) K^{\alpha-2}L^\beta & \alpha \beta K^{\alpha-1}L^{\alpha-1} \\
\alpha \beta K^{\alpha-1}L^{\alpha-1} & \beta (\beta - 1) K^\alpha L^{\beta-2}
\end{bmatrix}
\end{align*}
\]

Since \( 0 < \alpha < 1 \) we have \( \frac{\partial^2 Q}{\partial K^2} = \alpha (\alpha - 1) K^{\alpha-2}L^\beta < 0 \). Moreover

\[
\det H = [\alpha (\alpha - 1) \beta (\beta - 1) - \alpha^2 \beta^2] K^{2\alpha-2}L^{2\beta-2}
= [\alpha \beta (\alpha \beta - \alpha - \beta + 1) - \alpha^2 \beta^2] K^{2\alpha-2}L^{2\beta-2}
= [\alpha \beta (1 - \alpha - \beta)] K^{2\alpha-2}L^{2\beta-2} > 0
\]

Hence the function is concave.