1 Objective

We now turn to the main application of differentiation in economics: optimization problems. We first discuss the important distinction between relative and absolute maxima and minima. Then we discuss examples of optimization problems.

An appendix discusses the asymptotic behavior of rational functions

2 Absolute versus relative maxima

In the second handout of week 7 (sign diagrams) we introduced the following terminology which applies to every differentiable function $y(x)$:

a) A turning point is a critical point where the function turns from being increasing to being decreasing, i.e., where the first derivative switches sign.

Turning points come in two varieties:

a1) A peak (also called a relative or a local maximum) is a point where the function turns from being increasing to being decreasing or vice versa, i.e., where the first derivative changes sign from $+$ to $-$. 

a2) A trough (also called a relative or a local minimum) is a point where the function turns from being decreasing to being increasing, i.e., where the first derivative changes sign from $-$ to $+$. 

A relative maximum or minimum is by definition a critical or stationary point, i.e., it satisfies the equation $y'(x) = 0$. The conditions $y'(x) = 0$ for a critical point is often called the first order condition.

\[ y = -3x^4 + 28x^3 - 84x^2 + 96x - 32 \]

The function in the above graph has three turning points: two peaks at $x = 1$ and at $x = 4$ and a trough at $x = 2$. 

Fig. 1: $y(x) = -3x^4 + 28x^3 - 84x^2 + 96x - 32$
Theorem 1 Suppose $x_0$ is a critical point of the twice continuously differentiable function $y(x)$, i.e., $y'(x_0) = 0$. Then the following statements hold:

i) If $y''(x_0) < 0$ then $x_0$ is a relative maximum.

ii) If $y''(x_0) > 0$ then $x_0$ is a relative minimum.

iii) If $y''(x_0) = 0$ then $x_0$ can be either a relative maximum, a relative minimum or a saddle point.

Concerning iii) consider the following three functions at the critical point $x = 0$:

<table>
<thead>
<tr>
<th>$y(x)$</th>
<th>$y'(x)$</th>
<th>$y'(0)$</th>
<th>$y''(x)$</th>
<th>$y''(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3$</td>
<td>$3x^2$</td>
<td>$0$</td>
<td>$6x$</td>
<td>$0$</td>
</tr>
<tr>
<td>$x^4$</td>
<td>$4x^3$</td>
<td>$0$</td>
<td>$12x^2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$-x^4$</td>
<td>$-4x^3$</td>
<td>$0$</td>
<td>$-12x^2$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

They all satisfy $y'(x) = y''(x) = 0$, but at $x_0 = 0$ the first function has a saddle point the second a local minimum and the third a local maximum.

In cases like these a sign diagram for the second derivative is needed to determine which of the three types of a critical points is given.

Underlying parts i) and ii) is the following intuition: When $y''(x_0) < 0$, then, since the second derivative is assumed to be continuous, the second derivative must remain negative around $x_0$. Therefore the function is concave ($\sim$) around the critical point $x_0$ where it has a horizontal tangent. Hence only the shape of a relative maximum fits. Similarly $y''(x_0) > 0$ implies that the function is convex around $x_0$ and hence it must have a relative minimum at $x_0$.

The example in Figure 1 has the derivatives

$$y'(x) = -12x^3 + 84x^2 - 168x + 96$$
$$y''(x) = -36x^2 + 168x - 168$$

Trying the various factors of 96 we find that $+1$, $+2$ and $+4$ are critical points of the function. Since a cubic polynomial can have at most three roots there can be no further critical points.$^1$

$^1$A polynomial cannot have more roots than its degree. Every roots corresponds to a linear factor. In our case

$$y'(x) = -12(x-1)(x-2)(x-4)$$

must be the complete factorisation because an additional linear factor would give us a polynomial of degree 4.
Evaluating

\[
\begin{align*}
y''(1) &= -36 < 0 \\
y''(2) &= 24 > 0 \\
y''(4) &= -72 < 0
\end{align*}
\]

we find that the function has indeed relative maxima at \(x = 1\) and \(x = 4\) and a relative minimum at \(x = 2\).

3 Absolute maxima and minima

Suppose the function \(y(x)\) is defined on a set of numbers \(S\), typically the domain of the function or an interval like \(0 < x < 9\) or the set of all non-negative numbers \(0 \leq x\).

**Definition 2** A number \(x_0\) is called an absolute (or global) maximum of the function \(y(x)\) with respect to the set of numbers \(S\) if for all values of \(x\) in \(S\)

\[y(x_0) \geq y(x)\]

\(y(x_0)\) is then called the maximal value of the function \(y(x)\) on \(S\).

Absolute minima are defined correspondingly. There is still something relative about absolute maxima or minima, namely the reference to the set of numbers \(S\). Compare with the following statements: Pennsylvania Hill (relative maximum) is not the highest point in Europe, but Mont Blanc is (absolute maximum with respect to Europe). The highest point on Earth is Mount Everest (absolute maximum with respect to the world.)

The distinction between absolute and relative maxima is not always made clear in A-level courses, but it is important. Consider again the example in Figure 1. Suppose \(y(x)\) would be the profit function of a firm. Then profit is maximized at \(x = 4\) (the absolute maximum), not at \(x = 1\), which is only a relative maximum.

An absolute maximum is not necessarily a relative maximum. To see this consider the function \(y(x) = x\) on the interval \(0 \leq x \leq 1\).

![Graph](attachment:image.png)

Clearly, \(x = 1\) is an absolute maximum of the function although it is not a turning point.

A function does not necessarily have an absolute maximum or minimum. However, one has the following result.

\(^2\)However, it is not much slower to get these conclusion by using sign diagrams and the factorization

\[
y''(x) = -36 \left( x - \frac{7}{3} - \frac{\sqrt{7}}{3} \right) \left( x - \frac{7}{3} + \frac{\sqrt{7}}{3} \right)
\]
Theorem 3 Suppose a function is defined and continues on an interval $a \leq x \leq b$. Then it attains an absolute maximum and an absolute minimum in this interval.

Intervals of the form $a \leq x \leq b$ are called compact. The important properties of a compact interval are that it contains the two endpoints and that it is of finite length. On an interval of infinite length like $0 \leq x$ a function does not necessarily have an absolute maximum or minimum (take the function $y(x) = x$ for example). The following function

![Graph of $y(x) = \frac{x}{1-x^2}$](image)

is continuous on the interval $-1 < x < 1$ which misses the two endpoints. The function does not obtain a maximum or a minimum.

### 3.1 Finding an absolute maximum or minimum

Suppose the function $y(x)$ is twice continuously differentiable on the compact interval $a \leq x \leq b$. Then an absolute maximum or minimum with respect to this interval can be found as follows:

1. Determine all critical points of the function in the interval.

2. Calculate $y(x)$ for the two endpoints of the interval and for all critical points in between.

3. The value for which $y(x)$ is largest (smallest) is the absolute maximum (minimum).

For the function in Figure 1 and the interval $0 \leq x \leq 5$ we proceed for instance as follows. The critical points in the interval are 1, 2 and 4. Hence we calculate

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(x)$</td>
<td>-32</td>
<td>5</td>
<td>0</td>
<td>32</td>
<td>-27</td>
</tr>
</tbody>
</table>

We conclude that with respect to this interval the absolute minimum is at $x = 0$ and the absolute maximum at $x = 4$. 


3.2 Single-peaked functions

There is one frequently occurring case where the notions of absolute and relative maximum coincide. Namely, when the function is single-peaked in the sense that it has only one peak and no troughs. In such cases one can often apply the following result:

**Theorem 4** Suppose the twice-continuously differentiable function is defined in the interval $I$ and has one and only one critical point $x_0$ in this interval. If $y''(x_0) < 0$ then $x_0$ is an absolute maximum of the function on this interval.

4 Maximizing profits when marginal costs are increasing

We consider in this section a firm in a perfectly competitive market where many firms produce the same product. In such markets a single firm’s impact on the market price is negligible and it acts as a price taker, i.e., it takes the market price $P$ as a given fixed quantity which it cannot influence.

Assuming increasing marginal costs we will show that the individual supply curve of such a firm is its marginal costs curve and that the individual supply function is the inverse of the marginal cost function.

The total revenue of a firm is the product market price times the quantity $Q$ sold by the firm:

$$TR(Q) = P(Q)Q$$

The marginal revenue is the derivative of total revenue with respect to quantity,

$$MR(Q) = \frac{dTR}{dQ}$$

i.e., it is roughly the increase in total revenue when the firm produces a single (small) unit of output more.

$$MC'(Q) = \frac{dTC}{dQ} = 10Q + 20$$

A price taking firm will regard the price as a constant. Hence its marginal revenue is equal to the price: The price is fixed by the market, so an additional unit sold increases the revenue by the price $P$.

$$MR(Q) = \frac{dTR}{dQ} = P$$

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3 More precisely, that part of the supply curve where the firm produces a positive quantity.
If there is no uncertainty a firm will produce exactly what it wants to sell. Let $TC (Q)$ denote the total cost function of the firm, for instance

$$TC (Q) = 5Q^2 + 20Q + 110.$$ 

This was the second example in an earlier handout. The marginal cost function in this example is increasing. The marginal cost curve and the marginal cost function are given in the figure on the previous page. Recall that if the producer is currently producing the quantity $Q$, then it will cost him (roughly) the marginal costs $MC (Q)$ to produce a single (small) unit more. Recall also how you can read off this information from the graph: For a given quantity $Q$ on the horizontal axis move upwards to the point on the graph. The height of this point is the marginal cost.

The profit function of the firm is in general

$$\Pi (Q) = TR (Q) - TC (Q)$$

If the (absolute) profit maximum $Q^*$ is a critical point of the profit function (we will check this later) it must satisfy the first order condition

$$0 = \Pi' (Q^*) = MR (Q^*) - MC (Q^*)$$

so marginal revenue must equal marginal costs

$$MR (Q^*) = MC (Q^*)$$

In a perfectly competitive market this means that price must equal marginal costs.

$$P = MC (Q^*) \quad (1)$$

This is plausible: If the price were above the marginal costs, the producer could produce one unit more and thereby make a gain. If the price were below the marginal costs the producer could produce one unit less and thereby increase his profits. So, in optimum price must equal marginal costs.

Notice how you can use the marginal cost curve above to find the profit optimum: Starting with the market price $P$ on the vertical axis we look to the right until we hit the marginal cost curve and below we can read off how much the firm would produce in optimum. Hence we have found the supply curve of the firm: The graph tells us how much the firm would produce for any given price. However, Equation (1) gives us this information only indirectly namely for a given price we must first solve this equation for $Q^*$ to find the quantity supplied. The supply function $Q^S (P)$ which tells us for each given price how much the firm will produce is the inverse of the marginal cost function. By tradition one does not invert the graph but, in the case of demand- and supply functions, one draws the independent variable $P$ on the vertical axis and the dependent variable $Q$ on horizontal axis.

The marginal cost curve in our example is $MC (Q) = 10Q + 20$.

Price equals marginal costs means hence

$$P = 10Q^* + 20 \quad (2)$$
whereby $Q^*$ is the profit-maximizing quantity. For instance, if the market price is $P = 80$ we obtain from equation (2) the unique solution $Q^* = 6$. This reasoning works for every market price $P$. The equation that price must equal marginal costs has the unique solution

$$Q^S(P) = Q^* = \frac{P - 20}{10} \quad (3)$$

and this equation gives us the supply function of the firm.

The above arguments assumed that the profit maximizing quantity is given by the first order condition $P = MC(Q^*)$. Let us now discuss for an increasing marginal cost function when this is indeed the case.

1. Since marginal costs are increasing a horizontal line can intersect the marginal cost curve at most once. Hence, for any given price there can be at most one critical point.

2. The marginal cost curve is increasing and thence the derivative of the profit function

$$\Pi'(Q) = P - MC(Q)$$

is decreasing. Recall that a function is strictly concave if and only if its first derivative is decreasing (where the latter is reflected by having “almost everywhere” a negative second derivative). Hence the profit function is strictly concave. Therefore, if the first order condition $P = MC(Q^*)$ has a solution $Q^*$ it will be the unique critical point of the profit function and it will be an absolute maximum of the profit function. In the example this happens, for instance, when $P = 80$.

3. It is, however, possible that the first order condition has no solution. This can happen in two ways:

   (a) The market price is lower than the minimal marginal costs $MC(0)$ at 0 and hence lower than the marginal cost at any quantity. In this case the derivative of the profit function

   $$\Pi'(Q) = P - MC(Q)$$

   is always negative, which means that profit is always decreasing in quantity. Clearly, it is then optimal for the firm to produce zero output. In the above
example this happens when the price is below 20, for instance when $P = 10$:

![Graph](image)

Algebraically Equation then has a *negative* solution and the profit function has a single peak in the negative. It is then optimal for the firm to produce an output as close to this peak as possible, i.e., to produce zero.

(b) It does not happen in most examples, but a priori it is possible that the price is higher than the marginal costs could ever get. For this to happen the marginal cost curve would have to look like this:

![Graph](image)

$$MC(Q) = 150 - \frac{100}{Q+1}$$

For prices above 150 the profit function is always increasing. Because the price is always above the marginal costs it always pays to produce a unit more. The firm would like to supply an infinite amount at such prices. Mathematically, an absolute profit maximum does not exist. Economically, the assumption of a price-taking firm is no longer adequate at such prices. Firms cannot bring arbitrarily large quantities to the market without having an impact on the price.

5 **Maximizing profits when marginal costs are constant**

For a price taking firm one gets similarly extreme results as in the Case b) just discussed when the marginal costs are constant. For instance, in the first example of the earlier handout the total cost function was

$$TC(Q) = 90 + 20Q.$$
The marginal costs curve is constant at height 20.

\[ MC(Q) = 20 \]

The profit function is linear in \( Q \)

\[ \Pi(Q) = TR(Q) - TC(Q) = PQ - 90 - 20Q = (P - 20)Q - 90 \]

When the price is below the marginal costs, the profit function is decreasing and it is optimal to produce zero output. When the price is above the marginal costs, the profit function is increasing and it is optimal to produce an infinite amount. When the price is exactly equal to the marginal costs, the profit function is flat and any output is profit maximizing. One obtains the extreme case of a horizontal supply curve. A supply function does not exist. One speaks of an “infinitely elastic supply curve”. If all firms in the market have the same costs, the only equilibrium price would be \( P = 20 \). Because of the fixed costs all firms would make losses and would have to exit in the long run.

### 6 Monopoly

One gets less extreme results with constant marginal costs for models of imperfect competition. For instance, a monopolist (no competition) will take fully account of the fact that the quantity he sells has an effect on the market price. Suppose that he has the cost function

\[ TC(Q) = 90 + 20Q \]

while he faces the demand function

\[ Q = Q^D(P) = 10.40 - \frac{1}{50}P \]

which tells us the quantity demanded at every given price. Solving for \( P \)

\[
Q = 10.40 - \frac{1}{50}P \\
50Q = 520 - P \\
P + 50Q = 520 \\
P = 520 - 50Q
\]
we obtain the *inverse* demand function

\[ P = P(Q) = 520 - 50Q \]

which tells us the price the monopolist can achieve when he brings the quantity \( Q \) to the market.

The total revenue is now

\[ TR(Q) = P(Q)Q = (520 - 50Q)Q = 520Q - 50Q^2 \]

and his marginal revenue is no longer simply the price

\[ MR(Q) = \frac{dTR}{dQ} = 520 - 100Q \]

Equating marginal costs with marginal revenue gives

\[
\begin{align*}
520 - 100Q &= MR(Q) = MC(Q) = 20 \\
500 &= 100Q \\
Q &= 5
\end{align*}
\]

i.e., it is optimal for him to produce 5 units. One can verify that this quantity actually maximizes profits and that the monopolist can make positive profits.

In the figure on the right the profit-maximizing quantity is obtained as the intersection of the downward sloping marginal revenue curve and the horizontal marginal cost curve.

7 **U-shaped average variable costs**

The third example of a total cost function discussed in the first handout, week 6, was

\[ TC(Q) = 2Q^3 - 18Q^2 + 60Q + 50 \]

We want to know which quantity a profit-maximizing firm with this cost function should produce when the market is perfectly competitive and the given market price is \( P \).

It turns out that the answer to this question depends on the average variable costs (AVC) and the marginal costs (MC). Hence, we must first discuss how the average variable costs curve looks and how it relates to the marginal costs curve.
In our example, the fixed costs are $FC = 50$ and the variable costs are hence

$$VC(Q) = 2Q^3 - 18Q^2 + 60Q.$$  

Average costs are generally costs per item produced, so the average variable cost function is in our example

$$AVC(Q) = \frac{VC(Q)}{Q} = 2Q^2 - 18Q + 60.$$  

As the graph indicates, the AVC curve is U-shaped, i.e., it is strictly convex and has a unique absolute minimum at $Q_{Min} = 4.5$. The minimum average variable costs are calculated as

$$AVC_{Min} = AVC(4.5) = 19.5$$

To see algebraically that the AVC curve is indeed U-shaped with the describe properties we a) differentiate

$$AVC''(Q) = 4Q - 18,$$

b) solve the first order condition

$$AVC''(Q) = 4Q - 18 = 0 \quad \text{or} \quad Q = \frac{18}{4} = 4.5,$$

c) observe that there is a unique solution at 4.5,

d) differentiate again

$$AVC''''(Q) = 4 > 0$$

and observe hence that our function is indeed strictly convex. In particular, $Q_{Min} = 4.5$ is the absolute minimum.

Recall that the marginal costs are the derivative of the total or variable costs (the latter two differ only by a constant term). They are

$$MC(Q) = \frac{dTC}{dQ} = \frac{dVC}{dQ} = 6Q^2 - 36Q + 60$$

and are also U-shaped.
8 The relation between AVC, MC and supply

Whenever the AVC curve is U-shaped, i.e., strictly convex with a unique absolute minimum, the following applies:

1) The AVC curve and the MC curve intersect in two points, once on the vertical axis and one in the minimum of the AVC curve.
2) In the downward-sloping part of the AVC curve the MC curve is below the AVC curve, in the upward-sloping part it is above.
3) Above the AVC curve marginal costs are strictly increasing.

The following picture illustrates these facts in our example:

Moreover,

4) The individual supply curve is given by the part of the MC curve above the AVC curve. More precisely:
   A) When the price is below the minimum average variable costs, it is optimal for the firm not to produce any output.
   B) When the price is above the minimum average variable costs, it is optimal for the firm to produce a positive amount of output. Namely, it is optimal to produce the largest quantity for which the price equals the marginal costs.
   C) When the price is exactly equal to the minimum average variable costs, two quantities are optimal to produce, namely zero and the quantity which minimizes AVC.

Applied to our example this means the following:
At prices below 19.5 it is optimal to produce zero.
When the price is exactly 19.5, both \( Q = 0 \) and \( Q = 4.5 \) are optimal.
When the price is, for instance, \( P = 30 \) we must first solve the equation \( P = MC(Q) \) or

\[
30 = 6Q^2 - 36Q + 60
0 = 6Q^2 - 36Q + 30 = 6(Q^2 - 6Q + 5) = 6(Q - 1)(Q - 2)
\]

Here both \( Q = 1 \) and \( Q = 5 \) solve this equation. The larger of the two, \( Q = 5 \), is the profit maximizing quantity.

Using the general formula to solve quadratic equations one can obtain the supply
function explicitly as follows:

\[ P = 6Q^2 - 36Q + 60 \]
\[ 0 = 6Q^2 - 36Q + 60 - P \]
\[ 0 = Q^2 - 6Q + 10 - \frac{P}{6} \]

\[ Q_{1/2} = \frac{-6 \pm \sqrt{36 - 4\left(10 - \frac{P}{6}\right)}}{2} = -3\sqrt{\frac{36 - 4\left(10 - \frac{P}{6}\right)}{4}} \]

\[ = -3 \pm \sqrt{9 - 10 + \frac{P}{6}} \]

and, by taking the larger root, one obtains the supply function

\[ Q^s = 3 + \sqrt{\frac{P}{6} - 1} \]

valid for prices above 19.5.

**Remark 5** It holds as well that the average total cost curve intersects the marginal cost curve in its minimum.

### 8.1 Sketch of the argument

Read this section only if you like math!

Finally we indicate why the four facts stated above hold. For a more verbal presentation see Begg, Economics.

Variable costs are, by definition, the product of quantity and average variable costs:

\[ VC(Q) = Q \times AVC(Q) \]

We can differentiate this equation using the product rule and obtain

\[ MC(Q) = AVC(Q) + Q\frac{dAVC}{dQ} \]

From this equation we see that marginal costs are equal to average variable costs at the minimum of the AVC curve (since there \( \frac{dAVC}{dQ} = 0 \)), they are below the AVC curve when the latter is downward-sloped (\( \frac{dAVC}{dQ} < 0 \)) and above when the latter is upward sloped (\( \frac{dAVC}{dQ} > 0 \)).

Differentiating again gives

\[ \frac{dMC}{dQ} = \frac{dAVC}{dQ} + \frac{dAVC}{dQ} + Q\frac{d^2AVC}{dQ^2} = 2\frac{dAVC}{dQ} + Q\frac{d^2AVC}{dQ^2} \]

\[ \frac{dAVC}{dQ} = 0 \] except for the minimum.

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4 The AVC curve cannot have saddle points since it is assumed to be strictly convex. This rules out \( \frac{dAVC}{dQ} = 0 \) except for the minimum.
We have $Q > 0$ and, since the $AVC$ curve is strictly convex, $\frac{d^2AVC}{dQ^2} > 0$. In the upward-sloping part of the $AVC$ curve we have $\frac{dAVC}{dQ} > 0$ and get overall $\frac{dMC}{dQ} > 0$, i.e., the marginal cost curve is increasing above the $AVC$ curve.

To see that the $AVC$ curve and the $MC$ curve meet on the vertical axis one has to know the definition of the derivative as a limit of difference quotient (or “rates of change”). Actually, $AVC (Q) = \frac{VC(Q)}{Q} = \frac{VC(Q) - VC(0)}{Q - 0} = \lim_{Q \to 0} AVC (Q)$.

$(AVC (0)$ is, of course, not defined.)

We have shown the statements 1 - 3 above.

Concerning statement 4 I skip the very technical argument why an absolute profit maximum always exists when the $AVC$ curve is U-shaped. (Essentially one can show that the profit function must be decreasing for very large quantities.) Assuming it exists, it can either be at $Q = 0$ or it can be at a positive quantity. In the latter case it must be a “peak” and hence the first order condition $P = MC (Q)$ must be satisfied. It follows that the part of a supply curve where a strictly positive quantity is produced must be a part of the marginal cost curve.

When zero output is produced, only the fixed costs are to be paid: $\Pi (0) = -FC$. For $Q > 0$ we can rewrite the profit function as follows:

$$\Pi (Q) = PQ - VC (Q) - FC = PQ - Q \times AVC (Q) - FC$$

For prices below the minimum average variable costs $P - AVC (Q)$ is negative for all quantities $Q > 0$. Therefore $\Pi (Q) < -FC = \Pi (0)$ and it is optimal to produce zero. In words: one loses on average more on variable costs per item produced than one gains in revenues and hence it is better to produce nothing. (The fixed costs must be paid anyway.)

For prices $P > AVC^{Min}$ only the largest solution to the equation $P = MC (Q)$ gives a point on the $MC$ curve which is above the $AVC$ curve. For this solution $P = MC (Q) > AVC (Q)$ is satisfied and hence $\Pi (Q) > \Pi (0)$. For all other solutions $\Pi (Q) < \Pi (0)$. Hence this solution is the only candidate for the profit maximum. Since we assumed one, this must be it.

When $P = AVC^{Min}$ one has $P = MC (Q^{Min}) = AVC (Q^{Min})$. Hence $\Pi (0) = \Pi (Q^{Min})$. All other critical points of the profit function can be ruled out, so these two quantities must be optimal.