1 Generalities

Please take notes!!!

I want an active learning style with many exercises, so we have to write!
Stupid questions welcome.

There is a huge diversity of ability and background knowledge in this class. The lecture will have to cope with this. So will go over some quite basic stuff so that every one can pass, but there will also be sophisticated stuff for those who want to learn more.

times and rooms. The lecture will last at least eight weeks, but I would be willing to extend it a little if there is a demand.

Monday 2 - 4, SC 6 (MBA lecture room)
Tuesday 9-10, SC B, EXCEPT in weeks 5, 6, 7 where it will be Amory 316

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assessment:
summative: 2 hour exam in January
55% elementary stuff. 10% which requires a decent understanding of the module. 35% hard questions. I want everyone to pass!!
formative: weekly two sets of exercises, one set will be discussed in the Tuesday class, the other is homework.

Literature: See module outline. See also econphd.net for suitable lecture notes. I will use Simon and Blume for the first part (up to and including the Kuhn Tucker theorem) I will then discuss Chapters 11 and 12 and thereafter Chapter 9 (plus some background from the previous chapters) in the “Further mathematics” by Sydsæter et al.

The lecture was not meant to scare you off, but to give you some idea of the kind of problems we hope to be able to deal with BY THE END of this module and mention some new jargon and some basic ideas that thereby arise. In fact, we may not even come to something like the search model we discuss in more generality. What I wanted to introduce with that model is the idea of solving the problem backwards, from end to beginning.
We start now an overview where you will hear many concepts for the first time. Try to follow the flow and the basic ideas as far as possible. In the next weeks we will go slowly, one-by-one over the various ideas. At the end of the module, read the overview again and be impressed by how much more you understand now.

2 Overview on optimization

Aim: Find the best of a given set of options
To do so the decision maker has to have clear objectives
We will assume throughout that the decision maker has clear preferences which are described by a utility function (the objective function)

\[ u(x) \]

where \( x \) is an option or choice from a set of choices \( X \) and \( u(x) \) is a decimal number measuring how good the optimum is in comparison to other options.

The decision maker tries to find the option for which his utility is largest, i.e. he tries to find

\[ \max_{x \in X} u(x). \]

The number

\[ M = \max_{x \in X} u(x) \]

is called the maximal value (of the objective function).

An option \( x^* \in X \) with

\[ u(x^*) = \max_{x \in X} u(x) \]

is called a maximum. One often finds the notation

\[ x^* \in \arg \max_{x \in X} u(x) \]

(short for “argument maximus” or similar, it’s all Latin to me).

The optimization problem is to
- find out whether a maximum exists,
- to find one if it exists,
- to determine whether there are one or more maxima,

Instead of looking for the largest number, one can also look for the smallest number. However, since minimizing a function is the same as maximizing the negative of a function, i.e., since

\[ - \max_{x \in X} u(x) = \min_{x \in X} (-u(x)) \]

we will largely ignore minimization problems because they can be rewritten as maximization problems.
2.1 The case of finitely many options

Integers: 1, 2, 3, ... n-1, n, n+1, ...

Suppose there are finitely many options $x_1, x_2, \cdots, x_n$.

E.g. 5, 7, 6, 8, 3, 4, 7 (but the following discussion is only of interest is the list is really long).

Find the largest of these numbers!

Here is an algorithm (i.e. a step-by-step how-to-do guide) how to find the optimum.

We construct inductively a sequence $y_1, y_2, \cdots, y_n$ starting at the end.

Set $y_n = x_n$.
Suppose we have found $y_{k+1}$ with $1 \leq k < n$.
Then set

$$y_k = \max [x_k, y_{k+1}]$$

Once we have completed the search we get the optimum

$$y_1 = \max [x_1, x_2, \cdots, x_n]$$

In the example we get the sequence

$$
\begin{align*}
    y_7 &= 7 \\
    y_6 &= \max [4, 7] = 4 \\
    y_5 &= \max [3, 7] = 7 \\
    y_4 &= \max [8, 7] = 8 \\
    y_3 &= \max [6, 8] = 8 \\
    y_2 &= \max [7, 8] = 8 \\
    y_1 &= \max [5, 8] = 8
\end{align*}
$$

So the maximum of our list of numbers is 8.

If one has a univariate function $u(x)$ defined over an interval $[a, b]$, one can divide the interval into may small subintervals and choose the $x_i$ as the endpoints of the subintervals. One can then find an approximate maximum using $y_n = u(x_n)$ and $y_i = \max [u(x_i), y_{i+1}]$ for $i < n$. However, it is difficult to say how good the approximation is.

2.2 A search problem

The following example has a similar spirit to the above search algorithm in the way it proceeds from the end back to the beginning. We only have to do a bit of arithmetic, however, we need some ideas from statistics which makes it tricky. It belongs into the category of “stochastic optimization problems” which we will cover, if at all, at the very end.

John needs urgently a house. From day 0 to T he can visit one house every day and decide whether to buy it. House all have the same quality and have a value of 300,000 to him. House prices are uniformly distributed between 100,000 to 200,000. Because time is costly, a house that he buys tomorrow only has 80% of the value it has to him today. So his discount factor is 0.8)
What is the best search strategy for John, i.e. one that minimizes his ex-ante expected costs?

One can show that the optimal strategy for John is to have cutoff-prices $c_t$. He buys in period $t$ if he is offered a house with price $p_t \leq c_t$ and otherwise not.

In the last period $T$ he must buy ($c_T = 200,000$). The expected house price is 150,000. His expected value is $EV_T = 150,000$.

In period $T - 1$ he has to discount. His gain if he waits is

$$0.8 \times V_T = 0.8 \times 150,000 = 120,000$$

So he should wait if

$$300,000 - p_{T-1} < 0.8 \times 150,000$$

His cut-off level is the price where he is indifferent between buying and waiting. It is hence

$$c_{T-1} = 300,000 - 0.8 \times V_T$$
$$= 300,000 - 0.8 \times 150,000$$
$$= (2 - 0.8) \times 150,000$$
$$= 1.2 \times 150,000 = 180,000$$

Before he goes to see a house, the probability $q_{T-1}$ of seeing a house with value $c_{T-1} = 180,000$ or lower is

$$q_{T-1} = \text{prob} (p_{T-1} \leq c_{T-1}) = \frac{c_{T-1} - 100,000}{200000 - 100000} = 80\%$$

His expected payment if he buys conditional on not accepting prices above 180,000 is:

$$\frac{c_{T-1} + 100000}{2} = \frac{180000 + 100000}{2} = 140000$$

His expected value from buying is

$$E (V|B) = 300000 - \frac{c_{T-1} + 100000}{2} = 160,000$$

Overall, his optimal ex ante expected value before he starts searching on day T-1 is

$$EV_{T-1} = 0.8 \times 160000 + 0.2 \times 0.8 \times 150000 = 152,000$$

In day $T - 2$ he can get $0.8 \times 152,000$ by waiting. His cut-off level for period 2 is hence given by

$$300,000 - c_{T-2} = 0.8 \times 152,000$$
$$c_{T-2} = 300000 - 0.8 \times EV_{T-1}$$
$$= 300000 - 0.8 \times 152000$$
$$= 178,400$$
The probability of seeing a house with a price below the cut-off level is
\[ q_t = \text{prob}(p_t \leq c_t) = \frac{c_{T-1} - 100,000}{200000 - 100000} = \frac{178400 - 100000}{200000 - 100000} = 0.784 \]

The expected house price \textit{conditional} on buying is:
\[ \frac{c_{T-2} + 100000}{2} = \frac{178400 + 100000}{2} = 139200 \]

Now we can calculate his maximal expected value for the problem in period \( T - 2 \):
\[
EV_{T-2} = q_{T-2} \times \left( 300000 - \frac{c_{T-2} + 100000}{2} \right) + (1 - q_{T-2}) \times 0.8 \times EV_{T-1} \\
= 0.784 \times (300000 - 139200) + 0.216 \times (152000) \\
= 158900
\]

this procedure can be iterated to determine this optimal strategy for all periods. The cut off level for any period \( t < T \) is determined by:
\[ c_{t-1} = 300000 - 0.8 \times EV_t \]

and then the
\[
EV_{t-1} = \frac{c_{t-1} - 100,000}{200000 - 100000} \times \left( 300000 - \frac{c_{t-1} + 100000}{2} \right) + \frac{200000 - c_{t-1}}{200000 - 100000} \times 0.8 \times EV_t
\]

One can show that \( c_t \) is \textit{increasing over time} while \( V_t \) is decreasing. The less periods ahead, the higher the prices John must accept while the maximal utility John can expect is decreasing because there are less options left.

Exercise: What do we get for \( T - 3, T - 4 \)?

\section*{2.3 Dynamic Programming}

We now consider a non-stochastic optimal control problem.\textsuperscript{1} Again, the idea is calculate the maximal value backwards, starting with the last period. The difference is that now this value depends on a “state variable” which summarizes all that is relevant. In the example the state variable \( x_t \) is the amount of money John has in his bank. There are again finitely many periods \( t = 0, \ldots, T \). The highest utility John can get by acting optimally is denoted \( V_t(x_t) \) and is now a function of \( x_t \).

John’s decision variable / “control” variable \( u_t \) is the amount of money he takes out of his account for consumption in period \( y \). We do allow John to go into debt, but this will not happen because we will assume \( x_T = u_T \). In the last period John cannot go into debt and so it is clearly the best for him to consume all he has. (There is no future for John.) Moreover, we take the initial amount of money \( x_0 \) he has as given. \( x_0 \) is a given constant, not a variable.

\textsuperscript{1}The example is non-stochastic only because we ignore that future interest rates are uncertain. If we would bring that in, the model would be quite a bit more complicated.
We assume that John has a discount factor $0 < \beta < 1$ which has the same role as before. How money grows over time is given by the rule

$$x_{t+1} = g(t, u_t, x_t) = a_t (x_t - u_t)$$

where $a_t > 0$ is a given constant. $x_t - u_t$ is the savings at the end of period $t$, $x_{t+1}$ is his savings at the beginning of period $t + 1$. We assume that his total utility is the sum of the discounted streams of per-period utilities. For simplicity, the per-period utility is given by $\ln (u_t)$. Thus total utility is

$$U (\vec{x}, \vec{u}) = \sum_{t=0}^{T} f(t, u_t, x_t) = \sum_{t=0}^{T} \beta^t \ln (u_t)$$

In dynamic programming we start again at the end $t = T$.
The utility in last period is, since $x_T = u_T$,

$$V_T (x_T) = \ln (x_T)$$

In period $T - 1$: utility for last two periods is

$$\ln (u_{T-1}) + \beta V_{T-1} (x_{T-1})$$

The Bellmann equation in general relates the value function in period $t$ with that of period $t + 1$. It has the form

$$V_t (x_t) = \max_{u_t} [\ln (u_t) + \beta V_{t+1} (g(t, x_t, u_t))]$$

$$= \max_{u_t} [\ln (u_t) + \beta V_{t+1} (a_{t+1} (x_{t+1} - u_{t+1}))]$$

Let us write

$$W_t (u_t, x_t) = \ln (u_t) + \beta V_{t+1} (a_{t+1} (x_{t+1} - u_{t+1}))$$

Then

$$V_t (x_t) = \max_{u_t} W_t (u_t, x_t)$$

For the second last period we get

$$W_{T-1} (u_{T-1}, x_{T-1}) = \ln (u_{T-1}) + \beta \ln (a_{T-1} (x_{T-1} - u_{T-1}))$$

$$= \ln (u_{T-1}) + \beta \ln (x_{T-1} - u_{T-1}) + \beta \ln (a_T)$$

For given savings $x_{T-1}$ we can now find, using the first order condition (FOC), the optimal consumption $u_{T-1}$ for period $T - 1$:

$$\frac{\partial W_{T-1}}{\partial u_{T-1}} = \frac{1}{u_{T-1}} - \beta \frac{1}{x_{T-1} - u_{T-1}} = 0$$

$$x_{T-1} - u_{T-1} = \beta u_{T-1}$$

$$x_{T-1} = (1 + \beta) u_{T-1}$$

$$u_{T-1} = \frac{x_{T-1}}{1 + \beta}$$

$$x_T = a_{T-1} \left( \frac{1 + \beta}{1 + \beta} - \frac{1}{1 + \beta} \right) x_{T-1} = a_{T-1} \frac{\beta}{1 + \beta} x_{T-1}$$
Substituting this optimal consumption we get the optimal utility for period $T - 1$ as

$$V_{T-1}(x_{T-1}) = \ln \left( \frac{x_{T-1}}{1 + \beta} \right) + \beta \ln \left( \frac{a_{T-1} \beta}{1 + \beta} x_{T-1} \right)$$

$$= (1 + \beta) \ln (x_{T-1}) + \beta \ln (a_{T-1} a_T) - (1 + \beta) \ln (1 + \beta)$$

$$= (1 + \beta) \ln x_{T-1} + C_{T-1}$$

where $C_{T-1}$ is a constant that does not depend on $x_{T-1}$.

Next we calculate

$$W_{T-2}(u_{T-2}, x_{T-2})$$

$$= \ln (u_{T-2}) + \beta (1 + \beta) \ln (x_{T-1}) + \beta C_{T-1}$$

$$= \ln (u_{T-2}) + \beta (1 + \beta) \ln (a_{T-2} (x_{T-2} - u_{T-2})) + \beta C_{T-1}$$

$$= \ln (u_{T-2}) + \beta (1 + \beta) \ln (x_{T-2} - u_{T-2}) + C_{T-2}$$

For given $x_{t-2}$ we can now calculate the optimal consumption $x_{T-2}$ using the FOC

$$\frac{\partial W_{T-1}}{\partial u_{T-2}} = \frac{1}{u_{T-2}} - \frac{\beta (1 + \beta)}{x_{T-2} - u_{T-2}} = 0$$

$$\beta (1 + \beta) u_{T-2} = x_{T-2} - u_{T-2}$$

$$\beta (1 + \beta) (x_{T-2} - u_{T-2}) = x_{T-2}$$

$$u_{T-2} = \frac{x_{T-2}}{1 + \beta + \beta^2}$$

We get

$$x_{T-1} = a_{t-1} \frac{\beta + \beta^2}{1 + \beta + \beta^2} x_{T-2}$$

and hence

$$V_{T-2}(x_{T-2})$$

$$= \ln \left( \frac{x_{T-2}}{1 + \beta + \beta^2} \right) + \beta (1 + \beta) \ln \left( a_{T-2} \frac{\beta + \beta^2}{1 + \beta + \beta^2} x_{T-2} \right) + \beta C_{T-1}$$

$$= (1 + \beta + \beta^2) \ln (x_{T-2}) + C_{T-2}$$

Bow suppose that $V_{t+1}$ has the form

$$V_{t+1}(x_{t+1}) = \left( \sum_{\tau=0}^{T-(t+1)} \beta^\tau \right) \ln (x_{t+1}) + C_{t+1},$$

which have have just shown for $t = T, T - 1, T - 2$.

Then we obtain
\[ W_t(u_t, x_t) = \ln(u_t) + \beta V_{t+1}(x_{t+1}) \]
\[ = \ln(u_t) + \beta V_{t+1}(a_{t+1}(x_t - u_t)) \]
\[ = \ln(u_t) + \beta \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right) (a_{t+1}(x_t - u_t)) + \beta C_{t+1} \]
\[ = \ln(u_t) + \beta \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right) \ln(x_t - u_t) + \beta \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right) \ln(a_{t+1}) + \beta C_{t+1} \]
\[ = \ln(u_t) + \beta \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right) \ln(x_t - u_t) + C_t' \]

We find the optimal \( u_t \) as follows

\[ \frac{\partial W_t}{\partial u_t} = \frac{1}{u_t} - \beta \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right) \frac{1}{x_t - u_t} = 0 \]

\[ \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right) \beta u_t = x_t - u_t \]

\[ u_t = \frac{x_t}{\sum_{\tau=0}^{T-t-1} \beta^\tau} \]

\[ x_{t+1} = a_{t+1}(x_t - u_t) = a_{t+1} \sum_{\tau=0}^{T-t-1} \beta^\tau x_t = a_{t+1} \frac{\beta \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right)}{\sum_{\tau=0}^{T-t-1} \beta^\tau} x_t \]

\[ V_t(x_t) = \ln \left( \frac{x_t}{\sum_{\tau=0}^{T-t-1} \beta^\tau} \right) + \beta \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right) \ln \left( a_{t+1} \frac{\beta \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right)}{\sum_{\tau=0}^{T-t-1} \beta^\tau} x_t \right) + \beta C_{t+1} \]

\[ = \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right) \ln(x_t) + C_t \]

so the formula must hold for all \( t \).

Since \( x_0 \) is fixed we can now determine inductively starting from \( x_0 \) all \( x_t \) and \( u_t \).
(Now we move \textit{forward} in time!)

\[ u_0 = \frac{x_0}{\sum_{\tau=0}^{T-1} \beta^\tau} \]

\[ x_1 = a_0 \left( x_0 - \frac{x_0}{\sum_{\tau=0}^{T-1} \beta^\tau} \right) = a_0 \beta \frac{\sum_{\tau=0}^{T-1} \beta^\tau}{\sum_{\tau=0}^{T-1} \beta^\tau} x_0 \]

\[ u_1 = \frac{x_1}{\sum_{\tau=0}^{T-1} \beta^\tau} x_1 = a_0 \beta \frac{x_0}{\sum_{\tau=0}^{T} \beta^\tau} \]
Suppose we knew

\[ x_t = \left( \prod_{\tau=0}^{t-1} a_{\tau} \right) \beta^{t-1} \sum_{\tau=0}^{T-t} \beta^\tau x_0 \]

then

\[
\begin{align*}
    u_t &= \frac{x_t}{\sum_{\tau=0}^{T-t} \beta^\tau} = \left( \prod_{\tau=0}^{t-1} a_{\tau} \right) \beta^{t-1} \frac{x_0}{\sum_{\tau=0}^{T} \beta^\tau} \\
    x_{t+1} &= a_t \left( x_t - u_t \right) = a_t \left( \prod_{\tau=0}^{t-1} a_{\tau} \right) \beta^{t-1} \left( \frac{\sum_{\tau=0}^{T-t} \beta^\tau}{\sum_{\tau=0}^{T} \beta^\tau} - \frac{1}{\sum_{\tau=0}^{T} \beta^\tau} \right) x_0 \\
    u_{t+1} &= \frac{x_{t+1}}{\sum_{\tau=0}^{T-t-1} \beta^\tau} = \left( \prod_{\tau=0}^{t} a_{\tau} \right) \beta^t \frac{x_0}{\sum_{\tau=0}^{T} \beta^\tau}
\end{align*}
\]

We have now calculated all \( x_t \) and \( u_t \) in terms of \( x_0 \) and hence solved the problem. Notice that

\[
\begin{align*}
    \frac{x_{t+1}}{x_t} &= a_{t+1} \frac{\beta \left( \sum_{\tau=0}^{T-t-1} \beta^\tau \right)}{\sum_{\tau=0}^{T-t} \beta^\tau} \\
    \frac{u_{t+1}}{u_t} &= \frac{\sum_{\tau=0}^{T-t} \beta^\tau x_{t+1}}{\sum_{\tau=0}^{T-t-1} \beta^\tau} = \beta a_{t+1} \\
    u_{t+1} &= \beta a_{t+1} u_t
\end{align*}
\]

Important keywords were here:
- discrete time optimization problems
- dynamic programming
- Value function
- Bellmann equation

### 2.4 Constrained optimization and the Lagrangian

This approach transforms a constrained optimization problem into an unconstrained optimization problem and a pricing problem.

The objective function is

\[
U (\bar{\bar{x}}, \bar{\bar{u}}) = \sum_{t=0}^{T} f(t, u_t, x_t) = \sum_{t=0}^{T} \beta^t \ln (u_t)
\]

Set \( a_T := 1 \). The relevant constraints in this example are

\[
\begin{align*}
    g_t (u_t, x_t) &= a_t (x_t - u_t) - x_{t+1} = 0 \quad t = 0, ..., T-1 \\
    g_T (u_T, x_T) &= x_T - u_T = a_T (x_T - u_T) = 0
\end{align*}
\]
$x_0$ is fixed and not a variable. Set $a_T := 1$.

For each constraint we introduce a Lagrange multiplier $\lambda_t$. Intuitively, the Lagrange multipliers are prices that are imposed for slightly violating the constraints. We can now write down the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} f(t, u_t, x_t) + \sum_{t=0}^{T} \lambda_t g_t(u_t, x_t)$$

We seek a maximum of the Lagrangian. This gives the FOC

$$\frac{\partial \mathcal{L}}{\partial x_t} = \lambda_t a_t - \lambda_{t-1} = 0$$

for $t = 1, \ldots, T$

$$\frac{\partial \mathcal{L}}{\partial u_t} = \beta^t - \lambda_t a_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial u_T} = \beta^T - \lambda_T = 0$$

We obtain

$$\lambda_{t-1} = \lambda_t a_t$$

$$\lambda_t = \lambda_{t+1} a_{t+1}$$

$$\frac{u_{t+1} \beta^t}{u_t \beta^{t+1}} = \frac{\lambda_t a_t}{\lambda_{t+1} a_{t+1}} = a_t$$

$$u_t = \frac{u_{t+1}}{\beta a_t}$$

$$u_t = \frac{u_T}{\beta^{T-t} \prod_{\tau=t}^{T-1} a_\tau}$$

Since $x_T = u_T$ we obtain

$$u_t = \frac{x_T}{\beta^{T-t} \prod_{\tau=t}^{T-1} a_\tau}$$

From here we can use the constraints to obtain the same solution as before

### 3 Optimal control theory and the Hamiltonian

In essence this is the same as the Lagrangian approach, just better adapted to the problem at hand. A closer look shows that the Lagrangian is just the sum of the Hamiltonians and one gets the same FOC.

Rewrite as a difference equation:

$$x_{t+1} = a_t (x_t - u_t)$$

$$x_{t+1} - x_t = a_t (x_t - u_t) - x_t = \tilde{g}(t, u_t, x_t)$$
\[ U(\bar{x}, \bar{u}) = \sum_{t=0}^{T} f(t, u_t, x_t) = \sum_{t=0}^{T} \beta^t \ln(u_t) \]

Introduce the new “costate” variables \( q_t \) (this is basically \( \lambda_t \)).

The Hamiltonian is then defined as

\[ H(t, u_t, x_t, q_t) = f(t, u_t, q_t) + q_t \tilde{g}(t, u_t, x_t) \]

The maximum principle states

In an optimal solution \( u_t \) should maximize the Hamiltonian and the \( q_t \) should satisfy the difference equation

\[ q_t - q_{t-1} = -H_{x_t} \]

Moreover, \( q_T = 0 \)

In our example the Hamiltonian is

\[ H(t, u_t, x_t, q_t) = f(t, u_t, x_t) + q_t \tilde{g}(t, u_t, x_t) \]

\[ = \beta^t \ln(u_t) + q_t (a_t (x_t - u_t) - x_t) \]

\[ q_t - q_{t-1} = -q_t (a_t - 1) = q_t (1 - a_t) \]

\[ q_{t-1} = a_t q_t \]

\[ q_t = a_{t+1} q_{t+1} \]

\[ \frac{\beta^t}{u_t} - q_t a_t = 0 \]

\[ \frac{\beta^{t+1}}{u_{t+1}} = q_{t+1} a_{t+1} \]

\[ \frac{u_{t+1}}{u_t} = \frac{\beta^{t+1} q_t a_t}{\beta^t q_{t+1} a_{t+1}} = \beta a_t \]

\[ u_{t+1} = \beta a_t u_t \]

One sees that we get the same conditions as before.