The paper has 3 parts. Your marks on the first part will be rounded down to 55 marks. Your marks on the second part will be rounded down to 15 marks. Obviously, you cannot get over 100 marks overall. This examination is closed book and no materials are allowed. **Full work must be shown** on your script. **Please write legibly.**

Dear Student, this is a sample exam, which you receive together with detailed answers to make you familiar with the structure of the January exam. There will also be a mock exam to which you are expected to submit your answers in the lecture on January 12th 2009. A further important source of exercises are the class and home exercises. awhich
Part A  (You can gain no more than 55 marks on this part.)

Problem 1 (5 marks) Rewrite the first term such that it is defined for \( h = 0 \) and \( x > 0 \). Simplify the second.

\[
\frac{\sqrt{x + h} - \sqrt{x}}{(x + h) - x} \quad \left[ - (ab^3)^{-3} \left( a^b b^b \right)^2 \right]^3 (-8a)^{-3} / (- (8a)^6)^{-3}
\]

Problem 2 (5 marks) Solve

\[
\ln x^3 + \ln y^2 = \ln 2 + \ln 36 \\
\ln x + \ln y = \ln 6
\]

Problem 3 (10 marks) Consider the function

\[
y(x) = \frac{1}{1 + x^2}
\]

i) Calculate and draw a sign diagram for the first derivative. Where is the function increasing or decreasing? Are there any peaks or troughs? Does the function have a (global) maximum? Is the function quasi concave?

ii) Calculate and draw a sign diagram for the second derivative. Where is the function convex or concave? Are there any inflection points?

Problem 4 (5 marks) For the function

\[
y(x) = \frac{1}{3} x^3 - x
\]

find the (global) maxima and minima a) on the interval \([-2, 0]\) and b) on the interval \([-2, 2]\).

Problem 5 (5 marks) Find the equation of the tangent plane of

\[
 z(x, y) = x^2 y^2 
\]

at the point \((x^*, y^*, z^*) = (1, 4, z(1, 4))\).

Problem 6 (10 marks) Show that the function

\[
u(x, y) = \ln(5x + y) - 5(x + y)^2
\]

is concave.
**Problem 7** (10 marks) A consumer has the utility function

\[ u(x, y) = 5x^2 + 6xy + y^2 + 38x + 19y \]

a) Determine the marginal utility for the two commodities. Is more always better for the consumer?

b) The consumer has a budget of £90. A unit of the first commodity costs £10 and a unit of the second £20. Write down the budget equation.

c) The consumer wants to maximize his utility subject to his budget constraint. Write down the Lagrangian for this problem.

d) Calculate the first order conditions for a critical point of the Lagrangian.

e) Assume only the budget constraint binds. Derive a linear equation to be satisfied by a critical point that does not involve the Lagrange multiplier \( \lambda \) for the budget constraint.

f) Use the equation from e) and the budget equation to find the constrained optimum.

**Problem 8** (5 marks) Consider a price-taking firm with total cost function \( TC(Q, w) \) where \( w \) is the wage rate. Assuming \( \frac{\partial TC}{\partial w} > 0 \) and an interior equilibrium show that the profits of the profit maximizing firm decrease if the wage rate increases.

**Problem 9** (10 marks) Solve the problem

\[
\max_{u_t \in [0, +\infty]} \sum_{t=0}^{T} \left[-u_t + 2 \ln x_t\right], \quad x_{t+1} = u_t x_t \text{ for } t = 0, \ldots, T - 1, \text{ } x_0 \text{ given.}
\]

**Problem 10** (5 marks) Find a solution to the differential equations

a) \( \frac{dx}{dt} = xt, \; x(0) = 1 \)

b) \( \frac{d^2x}{dt^2} = x, \; x(0) = 0, \; x'(0) = 1 \)

**Problem 11** (10 marks) Solve the problem

\[
\max \int (1 - 4x - 2u^2) \; dt, \; \dot{x} = u, \; x(0) = 0, \; x(1) \text{ free.}
\]
**Part B**  (You can gain *no more than 15 marks* on this part.)
(2 question worth 15 marks each, to be written)

**Problem 12** (15 marks) For a consumer with the utility function

\[ u(x, y) = u(x, y) = 5x^2 + 6xy + y^2 + 38x + 18y \]

derive his demand function. What is the marginal increase in utility if the budget of the consumer is slightly increased?

**Problem 13** (15 marks) Solve the problem

\[ \max \sum_{t=0}^{\infty} \frac{1}{3^t} \ln (x_t - u_t) \text{ subject to } x_{t+1} = u_t, \, x_0 > 0, \, u_t > 0 \]

**Part C**

**Problem 14** (20 marks) Sketch the graph of the region \( A \) carved out by the two inequalities

\[
\begin{align*}
x^2 + y^2 & \leq 4 \\
y & \leq x \\
x, y & \geq 0
\end{align*}
\]

For any point \((a, b)\) with \(a, b \geq 0\) in the plane use the Lagrangian approach to determine the point closest to \((a, b)\) within or on the boundary of \(A\).

(There will be two more questions worth 20 marks each.)

(BEEM103 January 2009)  
(End of the exam paper.)