The paper has 3 parts. Your marks on the first part will be rounded down to 55 marks. Your marks on the second part will be rounded down to 15 marks. Obviously, you cannot get over 100 marks overall. This examination is closed book and no materials are allowed. **Full work must be shown** on your script. **Please write legibly.**

Submit your answers in the first lecture in January 12th, Room 004, in the Harrison Building.
Part A  (You can gain no more than 55 marks on this part.)

Problem 1  (5 marks) Simplify

\[
\frac{x\sqrt{y} - y\sqrt{x}}{x\sqrt{y} + y\sqrt{x}} \quad \frac{\sqrt[3]{a} \left((4a)^6\right)^{\frac{1}{12}} \sqrt[3]{a^3}}{(4a^5)^{\frac{1}{12}} \sqrt[3]{a}}
\]

Solution 1

\[
\frac{x\sqrt{y} - y\sqrt{x}}{x\sqrt{y} + y\sqrt{x}} = \frac{(x\sqrt{y} - y\sqrt{x}) (x\sqrt{y} - y\sqrt{x})}{x\sqrt{y} + y\sqrt{x}} = \frac{x^2 - 2xy\sqrt{y} + y^2}{x^2y - xy^2} = \frac{y - 2\sqrt{xy} + x}{x - y} = \frac{(\sqrt{x} - \sqrt{y})^2}{x - y}
\]

\[
\frac{\sqrt[3]{a} \left((4a)^6\right)^{\frac{1}{12}} \sqrt[3]{a^3}}{(4a^5)^{\frac{1}{12}} \sqrt[3]{a}} = \frac{\frac{1}{12} \frac{1}{12} a^{\frac{1}{2}} (4^6 a^6)^{\frac{1}{12}} a^{\frac{3}{4}}}{(4a^5)^{\frac{1}{12}} a^{\frac{1}{2}} a^{\frac{1}{2}}} = \frac{4^{\frac{5}{12}} a^{\frac{1}{12}} a^{\frac{1}{6}} a^{\frac{3}{4}}}{4^{\frac{1}{12}} a^{\frac{1}{2}} a^{\frac{1}{2}}} = \frac{4^{\frac{5}{12}} a^{1 + \frac{1}{12} + \frac{1}{4} - \frac{5}{12}}}{4^{\frac{1}{12}} a^{\frac{1}{2} + \frac{1}{2}}} = (2^5)^{\frac{5}{12}} a^{\frac{4 + 6 + 9 - 5 - 6}{12}} = 2^{\frac{5}{12}} a^{\frac{5}{12}} = \sqrt[12]{2^5} \sqrt[3]{a^2}
\]

Problem 2  (5 marks) Solve

\[
\ln x^2 - \ln y^2 = \ln 25 \\
x - y = 4
\]

Solution 2

\[
x^2/y^2 = 25 \\
x^2 = 25y^2 \\
(y + 4)^2 = 25y^2 \\
24y^2 - 8y - 16 = 0 \\
3y^2 - y - 2 = 0
\]

\[
y = \frac{+1 \pm \sqrt{(-1)^2 + 4 \times 3 \times 2}}{2 \times 3} = \frac{+1 \pm \sqrt{25}}{6} = \frac{+1 \pm 5}{6}
\]

Thus \(y = \frac{6}{6} = 1\) or \(y = -4/6 = -2/3\) and correspondingly \(x = 4 + y = 5\) or \(x = 3\frac{1}{3}\).

Problem 3  (10 marks) Consider the function

\[
y(x) = e^{-x^2-x^4}
\]

i) Calculate and draw a sign diagram for the first derivative. Where is the function increasing or decreasing. Are there any peaks or troughs? Does the function have a (global) maximum. Is the function quasi concave?
ii) Show that the second derivative contains the factor
\[(2x^2 - 1) (4x^4 + 6x^2 + 1)\]

Draw a sign diagram for the second derivative. Where is the function convex or concave. Are there any inflection points?

Solution 3

\[
y(x) = e^{-x^2 - x^4}
\]
\[
y'(x) = -2x (1 + 2x^2) e^{-x^2(1+x^2)}
\]
\[
y''(x) = 2 (2x^2 - 1) (4x^4 + 6x^2 + 1) e^{-x^2(1+x^2)}
\]

a) critical point zero, \(y'\) positive to the left, negative to the right of zero. Function therefore increasing to the left, decreasing to the right. critical point hence unique maximum. 
\(y(x) = g(h(x))\) where \(g(u) = e^u\) and \(h(x) = -x^2 - x^4\). As a monotone transformation of a concave function the function is quasi concave.

b) Second derivative is zero at \(x = \pm \frac{1}{\sqrt{2}}\), in between it is negative, outside positive. \(x = \pm \frac{1}{\sqrt{2}}\) are hence inflection points with the function concave in between the roots and convex outside.

Problem 4 (5 marks) For the function
\[y = (x - 2)^2\]
find the (global) maxima and minima a) on the interval \([-2, 0]\) and b) on the interval \([-2, 8]\).

Solution 4 \(y' = 2(x - 2)\), \(y'' = 2\). The function is decreasing for \(x \geq 2\) and increasing thereafter. On the interval \([-2, 0]\) the function has therefore its maximum at \(-2\) and its minimum at \(x = 0\). On the interval \([-2, 8]\) it has its minimum at \(x = 2\) and its maximum at \(x = 8\) because \(y(8) = (8 - 2)^2 = 26 > 16 = (-2 - 2)^2 = y(-2)\).
Problem 5 (5 marks) Find the equation of the tangent plane of
\[ z(x, y) = \frac{x - y}{x + y} \]
at the point \((x^*, y^*, z^*) = (2, 3, z(2, 3))\).

Solution 5 \(z^* = \frac{2 - 3}{2 + 3} = -\frac{1}{5}\)
\[
\frac{\partial z}{\partial x} = \frac{1 \times (x + y) - (x - y) \times 1}{(x + y)^2} = \frac{2y}{(x + y)^2}, \quad \frac{\partial z}{\partial x}|_{x=2,y=3} = \frac{2 \times 3}{5^2} = \frac{6}{25}
\]
\[
\frac{\partial z}{\partial y} = \frac{-1 \times (x + y) - (x - y) \times 1}{(x + y)^2} = \frac{-2x}{(x + y)^2}, \quad \frac{\partial z}{\partial y}|_{x=2,y=3} = -\frac{2 \times 2}{5^2} = -\frac{4}{25}
\]
The formula for the total differential at \(x = 2, y = 3\)
\[ dz = \frac{\partial z}{\partial x}|_{x=2,y=3} \, dx + \frac{\partial z}{\partial y}|_{x=2,y=3} \, dy \]
yields the formula for the tangent
\[
\left( z + \frac{1}{5} \right) = \frac{6}{25} (x - 2) + -\frac{4}{25} (y - 3)
\]
\[ z = \frac{6}{25} x - \frac{4}{25} y - \frac{1}{5} \]

Problem 6 (10 marks) Show that the function
\[ u(x, y) = \ln(x + 5y) + \ln(5x + y) \]
is concave.

Solution 6 We have
\[
\frac{\partial u}{\partial x} = \frac{1}{x + 5y} + \frac{5}{5x + y}
\]
\[
\frac{\partial u}{\partial y} = \frac{5}{x + 5y} + \frac{1}{5x + y}
\]
\[
\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x + 5y)^2} - \frac{25}{(5x + y)^2} < 0
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = -\frac{5}{(x + 5y)^2} - \frac{5}{(5x + y)^2} < 0
\]
\[
\frac{\partial^2 u}{\partial y^2} = -\frac{25}{(x + 5y)^2} - \frac{5}{(5x + y)^2} < 0
\]
Set \(a = (x + 5y)^{-1}\) and \(b = (5x + y)^{-1}\). The Hessian of \(u\) is
\[
H = \begin{bmatrix}
-a^2 - 25b^2 & -5a^2 - 5b^2 \\
-5a^2 - 5b^2 & -25a^2 - b^2
\end{bmatrix}
\]
and hence
\[
\det H = \left( a^2 + 25b^2 \right) \left( 25a^2 + b^2 \right) - 25 (a^2 + b^2)^2
= 25a^4 + 25^2a^2b^2 + a^2b^2 + 25b^4 - 25a^4 - 50a^2b^2 - 25b^4
= (25^2 + 1 - 50) a^2b^2 = 576a^2b^2 > 0
\]

We see that the leading principle minors of the Hessian have the right sign and so \( u \) is concave, as was to be proved.

**Problem 7** (10 marks) A consumer has the utility function
\[
u(x, y) = -4 (3x - 12)^2 - (2y - 10)^2
\]
a) Determine the marginal utility for the two commodities. Is more always better for the consumer?

b) The consumer has a budget of £40. A unit of the first commodity costs £10 and a unit of the second £5. Write down the budget equation.

c) The consumer wants to maximize his utility subject to his budget constraint. Write down the Lagrangian for this problem.

d) Calculate the first order conditions for a critical point of the Lagrangian.

e) Assume only the budget constraint binds. Derive a linear equation to be satisfied by a critical point that does not involve the Lagrange multiplier \( \lambda \) for the budget constraint.

f) Use the equation from e) and the budget equation to find the constrained optimum.

**Solution 7** a)
\[
\frac{\partial u}{\partial x} = -24 (3x - 12) > 0 \text{ only for } x < 4
\]
\[
\frac{\partial u}{\partial y} = -4 (2y - 10) > 0 \text{ only for } y < 5
\]
No, more is not always better. b)
\[
\mathcal{L} = -4 (3x - 12)^2 x - (2y - 10)^2 + \lambda (40 - 10x - 5y)
\]
c etc)
\[
\frac{\partial \mathcal{L}}{\partial x} = -24 (3x - 12) - 10\lambda = 0
\]
\[
\frac{\partial \mathcal{L}}{\partial y} = -4 (2y - 10) - 5\lambda = 0
\]
\[
-24 (3x - 12) = 10\lambda = -8 (2y - 10)
9 (x - 4) = 2 (y - 5)
9x - 2y = 36 - 10 = 26
\]
\[
-24 (3x - 12) = -8 (2y - 10)
40 - 10x - 5y = 0
\]
Solution is: \( x = \frac{42}{13}, y = \frac{20}{13} \)
**Problem 8** (5 marks) A monopolist with inverse demand function \( p = f(Q) \) and with total cost function \( TC(Q) \) has to pay an excise tax \( t \) to be subtracted from the price \( p \) paid by consumers. Assuming an interior equilibrium show that the profits of the profit maximizing firm decrease as the excise tax increases.

**Solution 8** The profit function is

\[
\Pi(Q, t) = Q(f(Q) - t) - TC(Q)
\]

and hence

\[
\frac{\partial \Pi}{\partial t} = -Q > 0
\]

By the envelope theorem a marginal increase in tax changes profit by \( \frac{\partial \Pi}{\partial t} = -Q < 0 \), so profits decrease.

**Problem 9** (10 marks) Solve the problem

\[
\max_{u_t \in [0,1]} \sum_{t=0}^{T} \sqrt{u_t x_t}, \quad x_{t+1} = (1 - u_t) x_t \quad \text{for} \quad t = 0, \ldots, T - 1, \quad x_0 > 0 \quad \text{given}.
\]

**Solution 9** The value function takes the form

\[
V_t(x_t) = \sqrt{c_t x_t}
\]

with \( c_T = 1 \) and \( c_t = 1 + c_{t+1} \) for \( 0 \leq t < T \). (At hindsight I realize that this means that \( c_t = T + 1 - t \), so the whole question is simpler than I thought.) This is clear for \( t = T \) since obviously \( \sqrt{u_T x_T} \) is maximized over all \( u_T \in [0,1] \) at \( u_T^* = 1 \). Suppose it holds for \( t + 1 \) with \( 0 \leq t < T \). Then we must find

\[
V_t(x_t) = \max_{u_t \in [0,1]} W_t(x_t, u_t) = \max_{u_t \in [0,1]} \left[ \sqrt{u_t x_t} + \sqrt{c_{t+1} (1 - u_t) x_t} \right]
\]

The FOC yields

\[
\frac{\partial W_t}{\partial u_t} = \frac{1}{2} \sqrt{x_t} - \frac{1}{2} \sqrt{\frac{c_{t+1} x_t}{1 - u_t}} = 0 \iff \sqrt{1 - u_t} = \sqrt{c_{t+1} u_t} \iff u_t^* = \frac{1}{1 + c_{t+1}}
\]

and so

\[
1 - u_t^* = \frac{c_{t+1}}{1 + c_{t+1}}
\]

\[
V_t(x_t) = W_t(x_t, u_t^*) = \sqrt{\frac{x_t}{1 + c_{t+1}}} + \sqrt{\frac{c_{t+1}^2 x_t}{1 + c_{t+1}}} = \frac{1 + c_{t+1}}{\sqrt{1 + c_{t+1}}} \sqrt{x_t}
\]

since \( c_t \) was defined as \( 1 + c_{t+1} \). We obtain

\[
u_t^* = \frac{1}{1 + c_{t+1}} = \frac{1}{1 + (T + 1 - (t + 1))} = \frac{1}{T + 1 - t}
\]

\[
x_t^* = (1 - u_t) x_t^* = \frac{c_{t+1}}{1 + c_{t+1}} x_t^* = \frac{T - t}{T + 1 - t}
\]

from which we can inductively infer all \( x_t^* \) starting with \( x_0 \).
Problem 10 (5 marks) Find a solution to the differential equation

\[ \frac{dx}{dt} = x(1 - x) = \frac{1}{\frac{x}{1-x} + \frac{1}{1-x}} \]

with \( x(0) = 1/2 \).

Solution 10

\[ \int \left( \frac{1}{x} + \frac{1}{1-x} \right) \, dx = \int dt \]

\[ \ln x - \ln (1-x) = t + C \]

\[ \ln \frac{x}{1-x} = t + C \]

\[ \frac{x}{1-x} = e^{t+C} \]

\[ x = e^{t+C} - e^{t+C}x \]

\[ (1 + e^{t+C})x = e^{t+C} \]

\[ x = \frac{e^{t+C}}{1 + e^{t+C}} \]

Since \( x(0) = \frac{e^C}{1 + e^C} = \frac{1}{2} \) we obtain \( 2e^C = 1 + e^C \), hence \( e^C = 1 \), hence \( C = 0 \). The solution is \( x(t) = \frac{e^t}{1+e^t} \).

Problem 11 (10 marks) Solve the problem

\[ \max \int \left( 1 - u^2(t) \right) \, dt, \quad \dot{x}(t) = x(t) + u(t), \quad x(0) = 1, \quad x(1) \text{ free.} \]

Solution 11 The Hamiltonian is

\[ H = (1 - u^2) + p(x + u) \]

The maximum of the Hamiltonian with respect to \( u \) is given by the FOC \( -2u + p = 0 \), so \( p = 2u \). The solution to \( \dot{p} = -H_x \) is \( \dot{p} = \frac{dp}{dt} = -p \). Thus

\[ \frac{dp}{p} = -dt \]

\[ \int \frac{1}{p} \, dp = - \int dt = -t + C_1 \]

\[ \ln p = -t + C_1 \]

\[ p = e^{-t+C_1} = C_2 e^{-t} \]

Because \( x(1) \) is free we must have \( p(1) = 0 \) and hence \( C_2 = 0 \). Thus \( p(t) = 0 \) and therefore \( p(t) = 0 \) for all \( t \). Since \( p = 2u \) we get \( u^*(t) = 0 \). So \( \dot{x}(t) = x(t) \) and \( x(0) = 1 \), which has the unique solution \( x^*(t) = e^t \).

Part B (You can gain no more than 15 marks on this part.)
Problem 12 (15 marks) For a consumer with the utility function

\[ u(x, y) = -(10 - x)^2 - (10 - y)^2 \]

derive his demand function. What is the marginal increase in utility if the budget of the consumer is slightly increased?

Solution 12 The Lagrangian is

\[ \mathcal{L} = -(10 - x)^2 - (10 - y)^2 + \lambda_1 (b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y \]

the FOC are

\[ \begin{align*}
2 (10 - x) - \lambda_1 p_x + \lambda_2 &= 0 \\
2 (10 - y) - \lambda_1 p_y + \lambda_3 &= 0
\end{align*} \]

It is clear that the consumer never wants to consume zero of one or both commodities unless the budget constraint is binding since one of the marginal utilities is strictly positive along the axes. We must distinguish four more remaining (assuming \( p_x, p_y, b > 0 \)):

1. No constraint binds. Then all Lagrange multipliers are zero and \( x^* = 10, \ y^* = 10 \). This is the case if \((10, 10)\) is affordable, i.e. \( 10p_x + 8p_y \leq b \). The consumer is satiated. an increase in his budget does not make him better off.

2. The non-negativity constraint \( x = 0 \) binds with the budget constraint. So \( \lambda_3 = 0 \) by complementarity. \( y^* = b/p_y, \lambda_1 = 2 (10 - b/p_y) / p_y \geq 0, \) so \( y^* = b/p_y \leq 10, \) the consumer is not able to buy his ideal amount of the \( y \) commodity. Moreover,

\[ \lambda_2 = 2 (10 - b/p_y) \frac{p_x}{p_y} - 20 \geq 0 \]

\[ \Leftrightarrow 10p_y^2 - 10p_x p_y + b p_x \leq 0 \]

The relative price of the first commodity is higher than the marginal rate of substitution, so that it is not worth to buy of it.

3. The non-negativity constraint \( y = 0 \) binds with the budget constraint. So \( \lambda_2 = 0 \) by complementarity. \( x^* = b/p_x, \lambda_1 = 2 (10 - x^*) / p_x \geq 0, \) so \( x^* = b/p_x \leq 10, \) the consumer is not able to buy his ideal amount of the \( y \) commodity. Moreover,

\[ \lambda_3 = 2 (10 - x^*) \frac{p_y}{p_x} - 20 \geq 0 \]

\[ \Leftrightarrow 10p_x^2 - 10p_x p_y + b p_y \leq 0 \]

The relative price of the second commodity is so high that it is not worth to buy of it.
4. Only the budget constraint binds. So $\lambda_2 = \lambda_3 = 0$. We must have $x^* \leq 10$ and $y^* \leq 10$ to have the Lagrange multiplier $\lambda_1$ non negative. We obtain

$$\begin{align*}
\frac{2(10-x)}{2(10-y)} &= \frac{p_x}{p_y} \\
p_x x + p_y y &= b
\end{align*}$$

$$
\begin{align*}
p_x (10-y) &= p_y (10-x) \\
p_y x - p_x y &= 10 (p_y - p_x) \\
p_y p_y x - p_y p_x y &= 10 p_y (p_y - p_x) \\
p_x p_x x + p_x p_y y &= p_x b \\
(p_x^2 + p_y^2) x &= 10 p_y^2 - 10 p_x p_y + p_x b
\end{align*}$$

The solution is

$$
\begin{align*}
x^* &= \frac{10 p_y^2 - 10 p_x p_y + p_x b}{(p_x^2 + p_y^2)} \\
y^* &= \frac{10 p_x^2 - 10 p_x p_y + p_y b}{(p_x^2 + p_y^2)}
\end{align*}$$

These are non-negative if we are not in cases 2) or 3)

Concerning the last question in the problem, let $u^* (p_x, p_y, b)$ be the optimal utility achievable for the given budget. Then, by the envelope theorem

$$\frac{\partial u^*}{\partial b} = \frac{\partial \mathcal{L}}{\partial b} = \lambda_1$$

so a marginal increase in the budget raises utility by the value of the Lagrange multiplier for the budget constraint. We obtain in the four cases from above:

1) $\lambda_1 = 0$

2) $\lambda_1 = \frac{2(10 - b/p_y)}{p_y}$

3) $\lambda_1 = \frac{2(10 - b/p_x)}{p_x}$

3) $\lambda_1 = \frac{2(10 - x^*)}{p_x} = \frac{2}{p_x} \left(10 - \frac{10 p_y^2 - 10 p_x p_y + p_x b}{(p_x^2 + p_y^2)}\right)$

**Problem 13** (15 marks) Solve the problem (with $0 < \beta < 1$)

$$\max_{t=0}^\infty \beta^t \ln (x_t - u_t)^2 \text{ subject to } x_{t+1} = u_t, \ x_0 > 0, \ u_t > 0$$

**Solution 13** $H = 2 \beta^t \ln (x - u) + p u$. FOC: $\frac{\partial H}{\partial u} = p_t - \frac{2 \beta^t}{x_t - u_t} = 0$ and $p_{t-1} = \frac{\partial H}{\partial x} = \frac{2 \beta^t}{x_t - u_t}$. Also the transversality condition $\lim_{t \to \infty} p_t (x_t - x_t^*) \geq 0$ must hold for any admissible
sequence \( x_t \) and we must have \( x_{t+1}^* = u_t^*, \ x_0^* = x_0 \). We obtain from the first two conditions \( p_t = p_{t-1} = p \), so, since \( x_{t+1}^* = u_t^* \)

\[
x_t^* - u_t^* = x_t^* - x_{t+1}^* = \frac{2\beta^t}{p}
\]

\[
x_{t+1}^* = x_t^* - \frac{2}{p}\beta^t = \ldots = x_0^* - \frac{2}{p} (\beta^0 + \beta^1 + \ldots + \beta^t)
\]

\[
x_0^* - \frac{2}{p} \left( \frac{1 - \beta^{t+1}}{1 - \beta} \right)
\]

One can choose an admissible sequence \( x_t > 0 \) with \( \lim x_t \) arbitrarily small. Transversality implies hence (since \( x_t^* > 0 \) must hold for all \( t \)) that \( \lim_{t \to \infty} p_t x_t^* = \lim_{t \to \infty} x_t^* = 0 \). Now

\[
x_t^* = x_0^* - \frac{2}{p} \left( \frac{1 - \beta^t}{1 - \beta} \right)
\]

and then

\[
x_t^* = x_0^* - \frac{x_0^*(1 - \beta)}{2} \left( \frac{1 - \beta^t}{1 - \beta} \right) = \frac{\beta^t}{2} x_0^*
\]

**Part C**

**Problem 14** (20 marks) Sketch the graph of the area \( A \) carved out by the two inequalities

\[
(x - 1)^2 + y^2 \leq 25
\]

\[
x^2 + (y - 1)^2 \leq 25
\]

For any point \((a, b)\) in the plane with non-negative coordinates use the Lagrangian approach to determine the point closest to \((a, b)\) within or on the boundary of \( A \). How many cases do we have to consider?

**Solution 14** The Lagrangian is

\[
\mathcal{L} = -(x - a)^2 - (y - b)^2 + \lambda_1 (25 - (x - 1)^2 - y^2) + \lambda_2 (25 - x^2 - (y - 1)^2)
\]

the FOC are

\[
\frac{\partial \mathcal{L}}{\partial x} = -2(x - a) - 2\lambda_1 (x - 1) - 2\lambda_2 x = 0
\]

\[
\frac{\partial \mathcal{L}}{\partial y} = -2(y - b) - 2\lambda_1 y - 2\lambda_2 (y - 1) = 0
\]

We have four cases: None of the constraints are binding, only one is binding, both are binding. To have both constraints binding
A) If no constraint is binding the FOC yield \( x^* = a, \ y^* = b \) and consequently \((a, b)\) must be in the region A, indicated by “I” in the following graph.

\[
\begin{align*}
A) & \quad x = a, \ y = b \\
\text{and consequently} & \quad (a; b) \text{ must be in the region A, indicated by “I” in the following graph.}
\end{align*}
\]

B) If both constraints are binding then

\[
\begin{align*}
x^2 + (y - 1)^2 & = (x - 1)^2 + y^2 = 25 \\
-2y + 1 & = -2x + 1 \\
y & = x \\
(x - 1)^2 + x^2 & = 25 \\
0 & = 2x^2 - 2x - 24 = 2(x - 3)(x + 4)
\end{align*}
\]

Thus the two possible solutions are \((x_1^*, y_1^*) = (-3, -3)\) and \((x_2^*, y_2^*) = (4, 4)\). In the first case we have the two FOC

\[
\begin{align*}
\frac{\partial L}{\partial x} & = -2(-3 - a) + 8\lambda_1 + 6\lambda_2 = 0 \\
\frac{\partial L}{\partial y} & = -2(-3 - b) + 6\lambda_1 + 8\lambda_2 = 0
\end{align*}
\]

which have the solution

\[
\begin{align*}
\lambda_1 & = \frac{3}{7} + \frac{3}{7}b - \frac{4}{7}a \\
\lambda_2 & = \frac{3}{7} + \frac{3}{7}a - \frac{4}{7}b
\end{align*}
\]

Since we must have \(\lambda_1, \lambda_2 \geq 0\) it follows that \((x_1^*, y_1^*) = (3, 3)\) is the solution if

\[
\begin{align*}
3b & \geq 3 + 4a \\
3a & \geq 3 + 4b
\end{align*}
\]
This gives area V in the above graph. In the second case we have the two FOC
\[
\frac{\partial L}{\partial x} = -2(4-a) - 6\lambda_1 (x - 1) - 8\lambda_2 x = 0
\]
\[
\frac{\partial L}{\partial y} = -2(4-b) - 2\lambda_1 y - 2\lambda_2 (y - 1) = 0
\]
which have the solution
\[
\lambda_1 = -\frac{4}{7} - \frac{3}{7}a + \frac{4}{7}b
\]
\[
\lambda_2 = -\frac{4}{7} - \frac{3}{7}b + \frac{4}{7}a
\]
Since we must have \(\lambda_1, \lambda_2 \geq 0\) it follows that \((x_1^*, y_1^*) = (3, 3)\) is the solution if
\[
4b \geq 4 + 3a
\]
\[
4a \geq 4 + 3b
\]
This gives area IV in the above graph.
B) Suppose only the constraint \(25 = (x - 1)^2 + y^2\) is binding. Hence \(\lambda_2 = 0\) by complementarity the FOC are
\[
\frac{\partial L}{\partial x} = -2(x - a) - 2\lambda_1 (x - 1) = 0
\]
\[
\frac{\partial L}{\partial y} = -2(y - b) - 2\lambda_1 y = 0
\]
Division yields
\[
\frac{x - a}{y - b} = \frac{x - 1}{y} \iff xy - ay = xy - bx + b \iff bx = (a - 1)y + b
\]
and therefore
\[
25 = \left(\frac{(a - 1)y + b}{b} - 1\right)^2 + y^2 = \left(\frac{(a - 1)^2 + b^2}{b^2}\right)y^2
\]
\[
y = \pm \frac{5b}{\sqrt{(a - 1)^2 + b^2}}
\]
\[
x - 1 = \frac{a - 1}{b} \frac{5b}{\sqrt{(a - 1)^2 + b^2}} = \pm \frac{5(a - 1)}{\sqrt{(a - 1)^2 + b^2}}
\]
The first order conditions can be rewritten as
\[
a - 1 = (1 + \lambda_1)(x - 1)
\]
\[
b = (1 + \lambda_1)y
\]
which, since \( \lambda_1 \geq 0 \) implies that \( (x - 1) \) and \( y \) are positive multiples of \( a - 1 \) and \( b \), so the optimum is

\[
x^* - 1 = \frac{5(a - 1)}{\sqrt{(a - 1)^2 + b^2}}, \quad y^* = \frac{5b}{\sqrt{(a - 1)^2 + b^2}}
\]

To be admissible the solution must satisfy

\[
25 = (x^* - 1)^2 + (y^*)^2
\]
\[
25 \leq (x^*)^2 + (y^* - 1)^2
\]

Subtracting from the inequality the equation above yields

\[
0 \leq 2x^* - 1 - 2y^* + 1 \iff y^* \leq x^*
\]

Hence \((x^*, y^*)\) lies above the diagonal \( y = x \). Since the vector \((a, b)\) is obtained from \((x^*, y^*)\) by multiplying with \((1 + \lambda_1) \geq 0\), it follows that \((a, b)\) must be in the area II in the graph.

C) Only the constraint \( 25 = x^2 + (y - 1)^2 \) is binding. Hence \( \lambda_3 = 0 \) by complementarity the FOC are

\[
\frac{\partial L}{\partial x} = -2(x - a) - 2\lambda_2 x = 0
\]
\[
\frac{\partial L}{\partial y} = -2(y - b) - 2\lambda_2 (y - 1) = 0
\]

Division yields

\[
\frac{x - a}{y - b} = \frac{x}{y - 1} \iff xy - ay - x + a = xy - bx \iff (b - 1) x = a (y - 1)
\]

and therefore

\[
25 = \left( \frac{a(y - 1)}{b - 1} \right)^2 + (y - 1)^2 = \left( \frac{a^2 + (b - 1)^2}{(b - 1)^2} \right) (y - 1)^2
\]
\[
y - 1 = \pm \frac{5(b - 1)}{\sqrt{a^2 + (b - 1)^2}}
\]
\[
x = \frac{a}{b - 1} \left( \pm \frac{5(b - 1)}{\sqrt{a^2 + (b - 1)^2}} \right) = \pm \frac{5a}{\sqrt{a^2 + (b - 1)^2}}
\]

It follows as above that

\[
x^* = \frac{5a}{\sqrt{a^2 + (b - 1)^2}} \quad y^* - 1 = \frac{5(b - 1)}{\sqrt{a^2 + (b - 1)^2}}
\]

and that \((a, b)\) must be in the area III in the graph.
Problem 15 (20 marks) Suppose that for a twice continuously differentiable univariate function $f(x)$ defined on an interval there is a unique point $x^*$ which solves the equation $f'(x^*) = 0$. Suppose $f''(x^*) < 0$. Sketch an argument why $x^*$ must be an absolute maximum of the function. Is $f(x)$ necessarily concave?

Solution 15 Because $f''(x)$ is continuous there is a neighborhood around $x^*$ where $f''(x) > 0$ holds. $f'(x)$ is decreasing near $x^*$. Hence $f'(x) > 0$ for all $x < x^*$ near to $x^*$. But then we must have $f'(x) > 0$ for ALL $x < x^*$ because otherwise there would be by the intermediate value theorem a $x < x^*$ with $f'(x) = 0$, contradicting the assumptions. Hence $f(x)$ is increasing to the left of $x^*$ and hence $f(x) < f(x^*)$. It follows by a symmetric argument that $f(x)$ is decreasing for $x > x^*$ and hence $f(x^*) > f(x)$ for all $x > x^*$. Thus $x^*$ is an absolute maximum. Problem 3 above shows that $f$ does not necessarily have to be concave.