The paper has 3 parts. Your marks on the first part will be rounded down to 55 marks. Your marks on the second part will be rounded down to 15 marks. Obviously, you cannot get over 100 marks overall. This examination is closed book and no materials are allowed. Full work must be shown on your script. Please write legibly.

Submit your answers in the first lecture in January 12th, Room 004, in the Harrison Building.
Part A  (You can gain no more than 55 marks on this part.)

Problem 1  (5 marks) Simplify
\[
\frac{x\sqrt{y} - y\sqrt{x}}{x\sqrt{y} + y\sqrt{x}} \quad \frac{(4a^6)^{\frac{1}{12}} \sqrt{a^3}}{(4a^5)^{\frac{1}{12}} \sqrt{a}}
\]

Problem 2  (5 marks) Solve
\[
\ln x^2 - \ln y^2 = \ln 25
\]
\[
x - y = 4
\]

Problem 3  (10 marks) Consider the function
\[
y(x) = e^{-x^2 - x^4}
\]

i) Calculate and draw a sign diagram for the first derivative. Where is the function increasing or decreasing? Are there any peaks or troughs? Does the function have a (global) maximum? Is the function quasi concave?

ii) Show that the second derivative contains the factor
\[
\left(2x^2 - 1\right) \left(4x^4 + 6x^2 + 1\right)
\]

Draw a sign diagram for the second derivative? Where is the function convex or concave? Are there any inflection points?

Problem 4  (5 marks) For the function
\[
y = (x - 2)^2
\]
find the (global) maxima and minima a) on the interval \([-2, 0]\) and b) on the interval \([-2, 8]\).

Problem 5  (5 marks) Find the equation of the tangent plane of
\[
z(x, y) = \frac{x - y}{x + y}
\]
at the point \((x^*, y^*, z^*) = (2, 3, z(2, 3))\).

Problem 6  (10 marks) Show that the function
\[
u(x, y) = \ln(x + 5y) + \ln(5x + y)
\]
is concave.

(BEEM103 January 2009)  (Please turn over.)
**Problem 7** (10 marks) A consumer has the utility function

\[ u(x, y) = -4 (3x - 12)^2 - (2y - 10)^2 \]

a) Determine the marginal utility for the two commodities. Is more always better for the consumer?

b) The consumer has a budget of £40. A unit of the first commodity costs £10 and a unit of the second £5. Write down the budget equation.

c) The consumer wants to maximize his utility subject to his budget constraint. Write down the Lagrangian for this problem.

d) Calculate the first order conditions for a critical point of the Lagrangian.

e) Assume only the budget constraint binds. Derive a linear equation to be satisfied by a critical point that does not involve the Lagrange multiplier \( \lambda \) for the budget constraint.

f) Use the equation from e) and the budget equation to find the constrained optimum.

**Problem 8** (5 marks) A monopolist with inverse demand function \( p = f(Q) \) and with total cost function \( TC(Q) \) has to pay an excise tax \( t \) to be subtracted from the price \( p \) paid by consumers. Assuming an interior equilibrium show that the profits of the profit maximizing firm decrease as the excise tax increases.

**Problem 9** (10 marks) Solve the problem

\[
\max_{u_t \in [0, 1]} \sum_{t=0}^{T} \sqrt{u_t x_t}, \ x_{t+1} = (1 - u_t) x_t \text{ for } t = 0, \ldots, T - 1, \ x_0 > 0 \text{ given.}
\]

**Problem 10** (5 marks) Find a solution to the differential equation

\[
\frac{dx}{dt} = x (1 - x) = \frac{1}{\frac{x}{x} + \frac{1}{1-x}}
\]

with \( x(0) = 1/2 \).

**Problem 11** (10 marks) Solve the problem

\[
\max \int (1 - u^2(t)) \ dt, \ \dot{x}(t) = x(t) + u(t), \ x(0) = 1, \ x(1) \text{ free.}
\]
**Part B** (You can gain *no more than 15 marks* on this part.)

**Problem 12** (15 marks) For a consumer with the utility function

\[ u(x, y) = -(10 - x)^2 - (10 - y)^2 \]

derive his demand function. What is the marginal increase in utility if the budget of the consumer is slightly increased?

**Problem 13** (15 marks) Solve the problem (with \(0 < \beta < 1\))

\[
\max \sum_{t=0}^{\infty} \beta^t \ln (x_t - u_t)^2 \quad \text{subject to } x_{t+1} = u_t, \ x_0 > 0, \ u_t > 0
\]

**Part C**

**Problem 14** (20 marks) Sketch the graph of the area \(A\) carved out by the two inequalities

\[
\begin{align*}
(x - 1)^2 + y^2 & \leq 25 \\
x^2 + (y - 1)^2 & \leq 25
\end{align*}
\]

For any point \((a, b)\) in the plane with non-negative coordinates use the Lagrangian approach to determine the point closest to \((a, b)\) within or on the boundary of \(C\). How many cases do we have to consider? (Hint: the area is symmetric to the \(45^\circ\) line. One can hence assume \(a \geq b\) and then conclude the arguments using symmetry.)

(two more questions worth 20 marks each, to be written)

(BEEM103 January 2009)  
(End of the exam paper.)