BEE3028 – Economic Issues	Dieter Balkenborg
Lecture	Departments of Economics
Moral Hazard	University of Exeter

1 The problem

We consider here a monopolistic and risk-neutral firm (the "principal") seeking to employ a risk-averse agent. The agent, if employed, would have to perform a task that would yield the revenue w^+ if successful and the revenue w^- if a failure, where $w^+ > w^-$. The agent is again risk averse with a utility function over wealth $x = u(w)$ satisfying the same assumptions as before. We assume that the agent has initial wealth zero and denote by U_0' his reservation utility if he does not get employed by the principal and seeks employment elsewhere.

In this handout we assume that the probability of the task being successful depends on the unobservable effort of the agent. Effort is costly to the agent. We assume that there are finitely many levels of effort $k = 0, 1, \ldots, K$. The higher the effort level the higher the cost a_k for effort. Thus $0 \le a_0 < a_1 < \ldots a_K$. The costs of effort must be subtracted from the agent's expected utility from wealth in order to obtain his overall utility. The probability of failure p_k decreases in the level of effort, so $1 \ge p_0 > p_1 > \ldots > p_K \ge 0$.

A wage contract would have to fix a pair of wages (X, Y) paid contingent of whether the task is a success or a failure. If the agent accepts the contract (X, Y) and decides to work with the effort level k his expected net utility will be

$$(1 - p_k)\, u(X) + p_k u(Y) - a_k$$

We describe contracts by pairs (x, y) giving the gross utilities $x = u(X)$ and $y = u(Y)$, so the above expected utility of the agent is written as

$$(1 - p_k)\, x + p_k y - a_k$$

and the corresponding expected profit for the principal is

$$(1 - p_k)\left(w^+ - f(x)\right) + p_k\left(w^- - f(y)\right). \tag{PR$_k$}$$

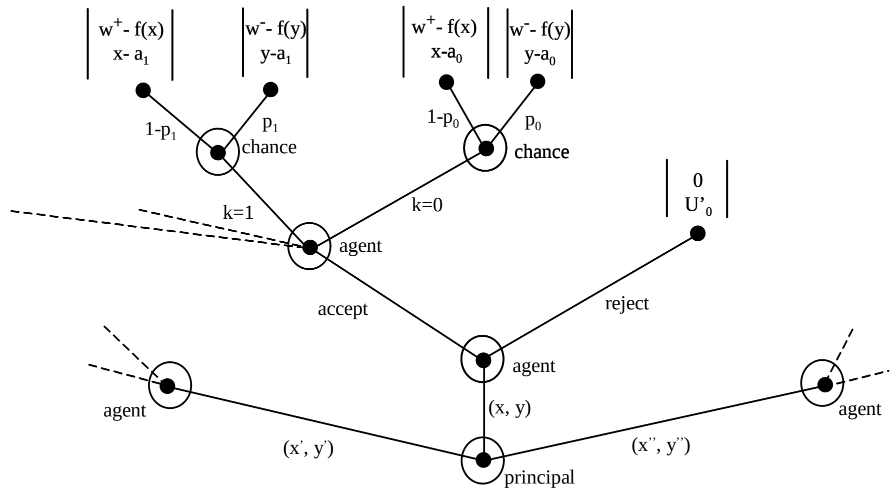
2 The extensive game

The timing of the model is as follows:

1. The principal offers a wage contract (x, y) to the agent.
2. Observing the offer, the agent can either accept or reject the offer. If he rejects, the agent gets the reservation utility U_0' and the principal the profit zero.
3. If the agent accepts, he must decide which effort level k to use.

4. A random move decides whether the task is a success or a failure with probability p_k. If it is a success, the agent's utility is $x - a_k$ and the principal's profit is $w^+ - f(x)$. If it is a failure, the agents utility is $y - a_k$ and the principal's profit $w^- - f(y)$

The following graph is an excerpt from the extensive game



3 Observable Effort

Recall first that with only one effort level $a_0 = 0$ a fixed wage rate independent of output is optimal. Let (X, Y) be the wage contract offered to the agent. X is the wage of the agent if (with probability $1-p$) the project is successful. Y is his wage if (with probability Y) the project is a failure.

Let (X^*, Y^*) be the wage contract which maximizes the principal's expected profit

$$(1-p)\left[w^+ - X\right] + p\left[w^- - WY\right]$$

subject to the agent's participation constraint

$$(1-p)\,u(X) + pu(Y) \geq U_0'$$

We claim that a fixed wage rate which is independent of whether the project is a success or a failure is optimal, i.e. that $X^* = Y^*$.

Suppose that instead $Y^* < X^*$.

Since the marginal utility of money for the risk-averse agent is strictly decreasing the marginal utility $u'(X^*)$ of the agent if the project is a success is *lower* than his marginal utility $u'(Y^*)$ when it is a failure.

Suppose the principal reduces the agent's wage if the project is a success by a fraction p of a pound while he increases it by a fraction $(1-p)$ when it is a failure.

The principal wins than the amount p with probability $(1-p)$ and looses the amount $1-p$ with probability p. His *expected* profit remains unchanged since $-(1-p)\,p + p\,(1-p) = 0$.

The marginal utility u' of the agent describes by how much his utility increases if he is given one pound more.

His utility if the project is a success is hence reduced by $pu'(X^*)$ units.

His utility if the project is a failure is hence increased by $(1-p)u'(Y^*)$ units.

Overall his expected utility changes by

$$-(1-p)pu'(X^*) + p(1-p)u'(Y^*) = p(1-p)(u'(Y^*) - u'(X^*)) > 0$$

The agent is made better of and hence certainly accepts the new contract. But this means the agent would still accept if he would be offered a contract which gives him a fraction p of a pound less than X^* if the project is a success and a little bit less than the fraction $(1-p)$ of a pound more if the project is a failure. Thus there is a contract which gives the principal a higher profit and which the agent would accept, a contradiction.

With several effort levels one can offer the agent a wage rate which is not contingent on output, but directly on effort. If the agent puts in the 'right' effort level a_k he gets a fixed wage rate X^* such that his utility from working $u(X^*) - a_k$ is equal to his reservation utility U_0'. Otherwise he gets zero. The effort level is chosen such that the principal's profit is maximized.

4 Solving the model

Following the backward induction procedure the optimal contract corresponding to a subgame perfect equilibrium can be found in the following two steps:

1. For each effort level k we have to find, if it exists, the profit maximizing contract induce this effort level, i.e., we have to solve the following constrained-optimization problem.

   Maximize the expected profit[1]

   $$(1-p_k)\left(w^+ - f(x)\right) + p_k\left(w^- - f(y)\right) \tag{Pr$_k$}$$

   subject to the participation constraint of the agent

   $$(1-p_k)x + p_k y - a_k \geq U_0' \tag{PC$_k$}$$

   and his incentive constraints

   $$(1-p_k)x + p_k y - a_k \geq (1-p_l)x + p_l y - a_l \tag{IC$_{kl}$}$$

   with respect to the other effort levels $l \neq k$.

   The participation constraint expresses that the agent is at least as well off by accepting the contract as by seeking employment elsewhere. The incentive constraints expresses that the agent is as well of by investing effort level k as by investing any other effort level.

[1] This formula underestimates the expected profit if the agent takes an even higher effort level than k.

2. Denote the solution to each of the above constrained optimization problems –if it exists – by $\left(x^{(k)}, y^{(k)}\right)$ and the expected profit it yields by Π_k. Set $\Pi_k = -\infty$ if no solution exists. Let k^* be the effort level for which Π_k is largest. If $\Pi_{k^*} \geq 0$ the principal offers the contract $\left(x^{(k^*)}, y^{(k^*)}\right)$. Otherwise he offers a contract like $(U_0' - 1, U_0' - 1)$ which the agent is certain to reject.

4.1 Solving the constrained optimization problems

Suppose the principal wants to implement effort level k in a profit-maximizing way. A contract inducing effort level k must satisfy the participation constraint (PC_k) and the incentive constraints (IC_{kl}) with $l \neq k$. The participation constraint can be written as[2]

$$y \geq \frac{U_0' + a_k}{p_k} - \frac{1 - p_k}{p_k} x$$

Thus the contract (x, y) has to lie above a line with slope $-\frac{1-p_k}{p_k}$ and intercept $\frac{U_0'+a_k}{p_k}$. An incentive constraints can be written as

$$(p_k - p_l)\, y \geq (p_k - p_l)\, x + a_k - a_l$$

Since the probability of failure is decreasing with the effort level and the cost of effort increasing we have $p_k < p_l$ and $a_k > a_l$ for $k > l$. When we divide the above inequality by the negative number $(p_k - p_l)$ the direction of the inequality changes and we get in this case

$$y \leq x + \frac{a_k - a_l}{p_k - p_l} \tag{1}$$

The constraint is binding along a line with slope $+1$ and a negative intercept $\frac{a_k - a_l}{p_k - p_l}$. A contract (x, y) inducing effort level k must be on or below this line. Similarly, we get in the case $l > k$ the condition

$$y \geq x + \frac{a_k - a_l}{p_k - p_l} \tag{2}$$

where the intercept is again negative since $a_k < a_l$.

Apart from two caveats the $K + 1$ constrained optimization problems hence have the following solutions:

1. The optimal way to induce effort level $k = 0$ is by the fixed-wage contract $(x^0, y^0) = (U_0' + a_0, U_0' + a_0)$

   This follows because all incentive constraints take the form (2) and are hence satisfied by (x^0, y^0). By Lemmas 1 and 2 from the handout on insurance the profit function (Pr_0) is maximized subject to the Participation Constraint (PC_0) at the contract (x^0, y^0).

2. For $k > 0$ let $l\,(k)$ be the effort level $0 \leq l < k$ for which $\frac{a_k - a_l}{p_k - p_l}$ is minimal. The optimal way to induce effort level k is then to offer the contract $\left(x^{(k)}, y^{(k)}\right)$ for

[2] I ignore the case $p_k = 0$ for which the arguments must be slightly adjusted.

which the Participation Constraint (PC_k) and the Incentive Constraint ($\text{IC}_{k,l(k)}$) are simultaneously binding. This yields two simultaneous equations with the solution

$$x^{(k)} = U_0' + a_k - p_k \frac{a_k - a_{l(k)}}{p_k - p_{l(k)}} \quad (3)$$

$$y^{(k)} = U_0' + a_k + (1 - p_k) \frac{a_k - a_{l(k)}}{p_k - p_{l(k)}} \quad (4)$$

The reasoning is as follows: For $0 < k < K$ the contracts which induce effort level k are in the strip below all the lines for which a Participation Constraints (IC_{kl}) with $l < k$ is binding and above all the lines for which a participation constraints (IC_{kl}) with $l > k$ is binding.[3] Since the Participation Constraint (PC_k) must also hold, it follows by Lemma 1 from the handout on insurance applied to the profit function (Pr_k) that the profit (Pr_k) is maximized in this strip where the participation constraint hold with equality. Applying Lemma 2 from the same handout implies that the incentive constraint ($\text{IC}_{k,l(k)}$) must be binding.

Remark 1 The first caveat to the above statement is that the "strip" we referred to can be empty because it could be the case for some $l_1 < k < l_2$ that $\frac{a_k - a_{l_1}}{p_k - p_{l_1}} < \frac{a_k - a_{l_2}}{p_k - p_{l_2}}$. This simply means that certain levels of effort cannot be induced because they are not cost-efficient. For instance if $a_k = 0.999$, $p_k = 0.5$ and $p_{k+1} = 0$, $a_{k+1} = 1$ it could easily be the case that the agent would always choose effort level $k + 1$ rather than k because the additional cost of effort is low while the reduction in the probability of failure is huge. This type of situation never arises if a_k depends on p_k via a decreasing and *convex* cost function $a_k = \phi(p_k)$. In this case one has always $l(k) = k - 1$.

Remark 2 The contract $\left(x^{(k)}, y^{(k)}\right)$ as calculated above may have a negative value for $y^{(k)}$. This is not allowed when we work with a utility function like $u(w) = \sqrt{w}$. In the latter case one must also consider the non-negativity constraint $y^{(k)} \geq 0$. When this non-negativity constraint rather than the participation constraint is binding the correct solution is $y^{(k)} = 0$, $x^{(k)} = -\frac{a_k - a_{l(k)}}{p_k - p_{l(k)}} > 0$.

Remark 3 In fact, to find the overall-optimal contract one can ignore all the incentive constraints (IC_{kl}) with $l > k$. More effort with the same contract cannot harm the principal.

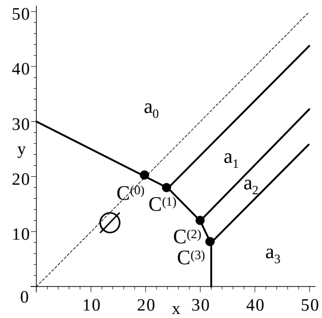
4.2 An example

Suppose $w^+ = 1600$, $w^- = 100$, $u(w) = \sqrt{w}$, $U_0' = 20$. There are three levels of effort

$k:$	0	1	2	3
$a_k:$	0	1	4	12
p_k	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	0

[3] For $k = K$ the last type of restrictions do not exist.

Our calculations will yield the conclusions illustrated in the following graph.



In this graph the region of contracts (x, y) which the agent would reject is indicated by "$\emptyset$". The region where a contract induces effort level k is marked by a_k. At the lines with slope 1 between the regions an incentive constraint is binding. The contract optimal to induce effort level k is indicated by $C^{(k)}$. With the above specifications it will turn out that overall the contract $C^{(3)}$ yields the highest expected profit to the principal.

We now calculate the contracts that are optimal to induce a given effort level.

The most profitable way to induce effort level 0 is given by the contract $x^{(0)} = y^{(0)} = 20$. It yields the expected profit

$$\Pi_0 = \frac{1}{3}\left(1600 - 20^2\right) + \frac{2}{3}\left(100 - 20^2\right) = 200 \qquad (\mathrm{Pr}_0)$$

The most profitable way to induce effort level 1 is given by the two equations

$$\begin{aligned} \frac{1}{2}x + \frac{1}{2}y &= 20 + 1 \Leftrightarrow x + y = 42 & (\mathrm{PC}_1) \\ \frac{1}{2}x + \frac{1}{2}y - 1 &= \frac{1}{3}x + \frac{2}{3}y \Leftrightarrow \frac{1}{6}(x - y) = 1 \Leftrightarrow x - y = 6 & (\mathrm{IC}_{1,0}) \end{aligned}$$

The solution is $x^{(1)} = 24$, $y^{(1)} = 18$. The corresponding expected profit is

$$\Pi_1 = \frac{1}{2}\left(1600 - 24^2\right) + \frac{1}{2}\left(100 - 18^2\right) = 400 \qquad (\mathrm{Pr}_1)$$

To induce effort level 2 we must have

$$\frac{2}{3}x + \frac{1}{3}y - 4 = 20 \Leftrightarrow 2x + y = 72 \qquad (\mathrm{PC}_2)$$

Among the participation constraints

$$\begin{aligned} \frac{2}{3}x + \frac{1}{3}y - 4 &\geq \frac{1}{3}x + \frac{2}{3}y \Leftrightarrow \frac{1}{3}(x - y) \geq 4 \Leftrightarrow x - y \geq 12 & (\mathrm{IC}_{2,0}) \\ \frac{2}{3}x + \frac{1}{3}y - 4 &\geq \frac{1}{2}x + \frac{1}{2}y - 1 \Leftrightarrow \frac{1}{6}(x - y) \geq 3 \Leftrightarrow x - y \geq 18 & (\mathrm{IC}_{2,1}) \end{aligned}$$

only the last one is relevant and must hold with equality. The solution to $2x + y = 72$ and $x - y = 18$ is $x^{(2)} = 30$, $y^{(2)} = 12$. The expected profit is

$$\Pi_2 = \frac{2}{3}\left(1600 - 30^2\right) + \frac{1}{3}\left(100 - 12^2\right) = 452 \qquad (\mathrm{Pr}_2)$$

To induce effort level 3 we have to consider the participation constraint

$$x - 12 = 20 \tag{PC$_4$}$$

and the incentive constraints

$$\begin{aligned} x - 12 &\geq \frac{1}{3}x - \frac{2}{3}y \Leftrightarrow \frac{2}{3}(x-y) \geq 12 \Leftrightarrow x - y \geq 18 && (\text{IC}_{3,0}) \\ x - 12 &\geq \frac{1}{2}x + \frac{1}{2}y - 1 \Leftrightarrow \frac{1}{2}(x-y) \geq 11 \Leftrightarrow x - y \geq 22 && (\text{IC}_{3,1}) \\ x - 12 &\geq \frac{2}{3}x + \frac{1}{3}y - 4 \Leftrightarrow \frac{1}{3}(x-y) \geq 8 \Leftrightarrow x - y \geq 24 && (\text{IC}_{3,2}) \end{aligned}$$

where the last one must be binding. The solution to $x - 12 = 20$ and $x - y = 24$ is $x^{(3)} = 32$ and $y^{(3)} = 8$. The corresponding expected profit is

$$\Pi_3 = 1600 - 32^2 = 576 \tag{Pr$_3$}$$

Overall it is optimal for the principal to propose the contract $(32, 8)$. The agent will then accept the contract, choose the highest effort level and get the expected utility 32. The principal will get the profit 576.

5 Exercises

Exercise 1 Use the above specifications except that the cost a_3 to induce the highest effort level are 14. How is the above solution affected? Which level of effort should the principal choose induce?

Exercise 2 With the specifications just given, suppose the agent actually owns the project and no principal is needed to conduct it. What level of effort would the agent invest?

Exercise 3 Suppose the level of effort of the agent would be observable and a contract can make the wage dependent on on both the effort and the outcome. If the principal has all the bargaining power, what contract should he offer to the agent to maximize his profit?

Exercise 4 Show: If the agent is risk-neutral the same outcome can be achieved in terms of expected utilities, regardless of whether effort is observable or not, by having the agent bear all the risk.