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1 Introduction

In later handouts we want to analyse some models with *asymmetric information*. These are situations where one party has relevant information that another party does not have. It gets strategically interesting if the uninformed party knows that the other party has the information and tries to deduce this information from the behaviour of the opponent. This may influence the behaviour of the informed party who may try to conceal her information. For instance, the owner of a large amount of shares in a firm may have learned that the firm is in difficulties. If he sells all his shares in one go, this will reduce the price and hence inform the market that the stock is bad which in turn will reduce the price even further...

Two basic types of asymmetric information have been studied extensively in the literature.

- **Adverse Selection,** called **Hidden Information** by Arrow. One party has information about the outcome of a random event which the other does not have. Typically: An employee knows whether he is very apt or not at a particular job. The employer does not have this information and hence a wage contract cannot condition on this information. Or: A car driver knows whether he is a "good driver" (low risk for an accident) or a "bad driver" (high risk for an accident), but he can do nothing about it. The insurance company does not know the "type" of the driver and hence cannot write contracts conditional on this information. The insurance company has the problem that an insurance contract that tries to attract the low-risks will primarily attract all the high-risks on which it may make losses. Hence the name "adverse selection".

- **Moral Hazard,** called **Hidden Action** by Arrow. One party has information about its own behaviour which is not observable by others. For instance, an employee may work hard or he may be lazy on his job and his effort is not per se observable to his employer. The wage contract can hence not condition on effort. Or: A car driver knows whether he drives risky or not. The insurer cannot observe this and hence an insurance contract cannot condition on this information. The problem is then that insurance may cause or at least exaggerate the very problem it is meant to cure. It creates a "moral hazard". The car driver would like to be insured against accidents. Once insured, however, his incentive to drive safely may be reduced. The disincentive created by insurance can be so strong that insurance may not be feasible.

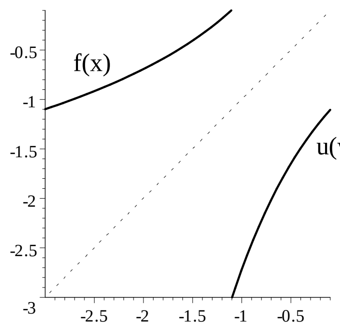
Our analysis will concentrate on the insurance model as primary example, you find many other examples in the textbooks. We will use the contract-theoretic approach to analyse the model. It is important in this approach that contracts are assumed to be binding. This approach is slightly "anti game-theoretical" insofar as it is not explicitly modelled why contracts are a commitment device. Rather, a functioning court system is implicitly assumed which ensures that breach of a contract will not occur because it is severely punished. Due to this assumption – and the assumption that the uninformed party can propose a contract as a take-it-or leave-it-offer – all lack of information can be reflected in the contracts and we obtain in the end models with perfect information which can in essence be solved using backward induction.

Before we analyse problems with asymmetric information we must first understand how to determine optimal insurance contracts when information is symmetric. This is the content of this handout.

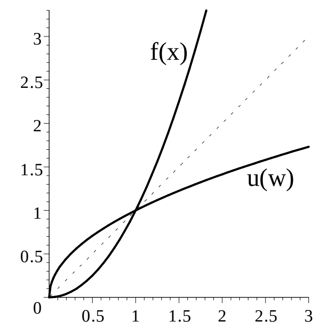
2 Insurance

An individual (the *agent*) faces the risk of an accident. Suppose that an accident occurs with a probability $0 < p < 1$. An accident would reduce the agent's wealth from w^+ to w^- where $w^- < w^+$. The agent seeks insurance from an insurer (called the *principal*). For the purpose of illustration we use $w^+ = \pounds 1600$ and $w^- = \pounds 100$, thus the cost of an accident is $\pounds 1500$. We will compare our results for the low, medium or high accident probabilities $p = p_L = \frac{1}{3}$, $p = p_M = \frac{1}{2}$ or $p = p_H = \frac{2}{3}$.

Since risk is involved the preferences of the individuals over lotteries matter. We will assume that the principal is *risk neutral*, i.e. he maximizes expected monetary gains (profits). In contrast, the agent will be assumed to be *risk averse*. He has a von Neumann - Morgenstern utility function $u(w)$ over wealth which is increasing (more money is better) and strictly convex. This means that marginal utility u' is decreasing, getting an additional pound increases his utility more when he has £10 than when he has £1.000.000. More precisely, his utility function $u(w)$ is assumed to be twice differentiable with $u'(w) > 0$ and $u''(w) < 0$ for all admissible w. The domain of the function $u(w)$ has to be some interval containing the numbers w^- and w^+. For simplicity we assume that the domain consists either of all numbers, as, for instance, when the utility function is $u(w) = -e^{-w}$ and exhibits constant absolute risk aversion, or it consists of all non-negative numbers, as, for instance, when the utility function is $u(w) = \sqrt{w}$ and exhibits constant relative risk aversion. The utility function $u(w) = \sqrt{w}$ will be frequently used for illustration.



$u(w) = -e^{-w}$; $f(x) = -\ln(-x)$



$u(w) = \sqrt{w}$; $f(x) = x^2$

Because the utility function $x = u(w)$ is increasing it has an inverse function $w = f(x)$, i.e. a function f such that $x = u(w)$ holds if and only if $w = f(x)$ holds. For instance, $x = -e^{-w}$ has the inverse function $w = -\ln(-x)$ and $x = \sqrt{w}$ has the inverse function $w = x^2$. As illustrated for these two functions in the above figures, the graph of the inverse function is obtained from the graph of the utility function by reflecting it at the 45°-line. The inverse function describes how much wealth w the agent must be given such that his utility is x. The derivative $\frac{dx}{dw} = u'(x)$ of the utility function u describes the *marginal utility* for money of the agent. It is decreasing $(u' < 0)$ because the agent is assumed to be risk averse. The derivative $\frac{dw}{dx} = f'(x)$ of the inverse f describes the *marginal monetary cost* of increasing the agent's utility by one unit. It is increasing, i.e. $f''(x) > 0$ and so $f(x)$ is a convex function.[1]

In the following we assume that the principal knows precisely the utility function of the agent.[2] This and the probability of an accident are assumed in this handout to be common knowledge between the principal and the agent. In the handout on adverse selection we will relax this assumption and assume that the principal does – in contrast to the agent – not know the accident probability. In the handout on moral hazard we will assume that the agent can affect the accident probability by his unobservable choice of effort.

Due to the assumptions that the principal, who is not facing a risk, is risk neutral and that the agent, who is facing a risk, dislikes risk there is an incentive to trade. Insurance contracts where the principal bears part of the risk can make both parties better off.

The main result of this handout is that it is under symmetric information optimal for the principal to offer full insurance. Our derivation of this result will presumably appear

[1] We have $x = u(f(x))$. The chain rule yields $1 = u'(f(x))\, f'(x)$ or $f'(x) = \frac{1}{u'(f(x))} > 0$. Differentiating again the chain rule and the quotient rule yields $0 = u''(f(x))\,(f'(x))^2 + u'(f(x))\, f''(x)$ or

$$f''(x) = -\frac{u''(f(x))}{u'(f(x))}\,(f'(x))^2 = -\frac{u''(f(x))}{(u'(f(x)))^3} > 0$$

using again $f'(x) = \frac{1}{u'(f(x))}$.

[2] This is implied by the assumption of having a game with complete information which requires the rules of the game including the payoffs to be common knowledge.

to be unnecessarily detailed. However, we want to discuss here the techniques which we must take for granted when discussing models of asymmetric information.

If the agent does not buy insurance his expected utility is

$$U_0 = (1-p)\, u\left(w^+\right) + pu\left(w^-\right) = (1-p)\, x_0 + py_0 \tag{1}$$

where $x_0 = u(w^+)$ and $y_0 = u(w^-)$ are the utilities he gets conditional on whether an accident occurs or not. With our numeric specifications the net utility of an uninsured agent if is $u(w^+) = \sqrt{1600} = 40$ if no accident occurs and $u(w^-) = \sqrt{100} = 10$ if an accident occurs. Thus $U_0 = 40\,(1-p) + 10p = 40 - 30p$. In particular, an uninsured agent with the low accident probability $p = \frac{1}{3}$ has the expected utility $U_0 = 30$, the one with the medium accident probability $p = \frac{1}{2}$ has the expected utility $U_0 = 25$ and the one with the high accident probability $p = \frac{2}{3}$ has the expected utility $U_0 = 20$. This will be the *reservation utility* the agent will use for comparison when he considers to buy insurance.

An *insurance contract* has to fix a *premium* P which the agent has to pay to the principal when he signs a contract and a (partial) *compensation payment* C from the principal to the agent when an accident occurs. Briefly an insurance contract is a pair (P, C) fixing these two numbers.

However, for the mathematical analysis it will be much easier to think of a contract in terms of the state-contingent utilities it generates for the agent. For a given contract (P, C) let x be the net utility which the agent receives in the state where no accident occurs and let y be his net utility in the state where an an accident occurs.

If no accident occurs the agent has the wealth w^+ from which the premium P is to be subtracted. His net utility x is hence

$$x = u\left(w^+ - P\right) \Leftrightarrow f(x) = w^+ - P \Leftrightarrow P = w^+ - f(x)$$

and the principal makes the profit

$$P = w^+ - f(x) \qquad \left(= 1600 - x^2\right)$$

from the contract. If an accident occurs the agent has already paid P, in addition his wealth is reduced to w^-, but he gets the partial compensation C. His net utility y is therefore

$$\begin{aligned} y &= u\left(w^- - P + C\right) \Leftrightarrow f(y) = w^- - P + C \\ &\Leftrightarrow C = -w^- + f(y) + P = w^+ - w^- - (f(x) - f(y)) \end{aligned}$$

and the principal makes the loss $C - P$, i.e. his profit is

$$P - C = w^- - f(y) \qquad \left(= 100 - y^2\right)$$

We see that every pair of state-contingent utilities (x, y) determines uniquely the premium P and the compensation payment C of the insurance contract and vice versa. It also determines the profits of the principal. Using in the following the induced pair of state-contingent utilities (x, y) rather than the pair (P, C) to describe the different insurance contracts has several advantages. First of all, for the game-theoretic analysis the utilities induced by a contract are important and not which numbers appear in the contract. Secondly, finding the insurance contract which is optimal for the principal will lead us to a constrained optimization problem that has linear constraints in (x, y), but not in (P, C). This will be convenient for the mathematical analysis and for drawing the graphs. Mathematically, all we are doing is a clever substitution of variables. Alternatively, instead of using the induced pairs of state-contingent utilities (x, y) to describe a contract we could also describe the contract by the induced pair $(X, Y) = (f(x), f(y))$ of final wealth the agent would end up with. This approach is used in most textbooks. In terms of the variables X and Y one obtains optimization problems with a linear objective function and non-linear constraints. When several constraints are involved, as in the problem with adverse selection, I find it easier to understand the state-space diagrams with utility instead of wealth on the axis because the various constraints are linear. You can judge for yourself by comparing the graphs in the textbooks with mine in the handout on adverse selection.

If $x = y$ the agent is equally well-off regardless of whether an accident occurs or not. Contracts with $x = y$ are said to provide *full insurance*. If a contract provides full insurance then the compensation C is equal to the damage costs $w^+ - w^-$ ($= £1500$). Full-insurance contracts may, however, differ in the premia they require. Given our numeric specifications consider, for instance, the contract with the premium $P = £975$ and the compensation $C = £1500$. Then the agent ends up with the wealth $£1600 - £975 = £625$ if no accident occurs and with $£100 + £1500 - £975 = £625$ if an accident occurs. His utility is therefore $x = y = \sqrt{625} = 25$ regardless of whether an accident occurs or not. As we have seen above, the agent with the high accident probability $p = \frac{2}{3}$ has the reservation utility $U_0 = 20$. He would accept this contract and make a gain in expected utility. The agent with the medium accident probability $p = \frac{1}{2}$ has the reservation utility $U_0 = 25$ and is hence indifferent between accepting the contract and remaining uninsured. The agent with the low accident probability $p = \frac{2}{3}$ has the reservation utility $U_0 = 30$. He would not accept the contract although it offers full insurance because the premium is too high. He would not accept a full-insurance contract with a premium higher than £700. (Why?)

The insurance-contract model is very similar to an employment-contract model where the principal tries to hire the agent in order to perform a risky task for him. In the employment-contract model w^+ would be the revenue the task generates if it is a success whereas w^- would be the revenue generated by the task if it were a failure (or, at least, less successful). The employment contract could condition the wage paid to the agent on whether the task is a success or not. Assuming the initial wealth of the agent to be zero, the employment contract (x, y) would specify the wage $f(x)$ to be paid to the agent if the task is a success and the wage $f(y)$ if it is a failure. Again, the wage contract is completely determined by the pair (x, y). The agent is said to be paid a *fixed wage rate* if his wage is independent of whether the task is a success or not, i.e. if $f(x) = f(y)$ and hence $x = y$.

The only difference between the insurance-contract model and the employment-contract model concerns what happens when no contract is made. In the insurance-contract model the agent is then fully exposed to the risk of an accident and his expected utility is the number U_0 given by the formula (1) above. In the employment model, when the agent does not accept an employment contract, the task is not undertaken and he will have to seek employment elsewhere. His expected utility from seeking employment elsewhere will be some reservation utility U_0' which is not necessarily related to U_0 as given by the formula (1) above.

2.1 The contract curve

Before continuing, we calculate here the set of Pareto-efficient allocations in two examples. A contract (P, C) is in this subsection described by the net wealth (X, Y) Ann receives conditional on whether an accident occurs or not.

See Varian, "Intermediate Microeconomics", Chapter 30 for the Edgeworth Box and the contract curve.

Problem Ann faces a risk where her wealth may be reduced from £1600 (state A) to £100 (state B) with a probability of 50%. Bob faces no risk, he owns £1000 in both states. However, he can trade risk with Ann. The result of a possible trade is described by a pair (X, Y) where Ann gets X pounds in state A and Y pounds in state B. Since the total wealth is £2600 in state A and £1100 in state B, this means that Bob owns £$2600-x$ in state A and £$1100-y$ in state B. Both have preferences over lotteries satisfying the von-Neumann Morgenstern axioms. Ann's preference is described by the utility function for money $u(w) = \sqrt{w}$. Thus her expected utility from the trade (x, y) is $\frac{1}{2}\sqrt{x} + \frac{1}{2}\sqrt{y}$

a) (6 marks) Suppose Bob's von-Neumann Morgenstern utility function is also $v(w) = \sqrt{w}$. His expected utility from the trade (x, y) is hence $\frac{1}{2}\sqrt{2600 - x} + \frac{1}{2}\sqrt{1100 - y}$. Sketch the Edgeworth box and the indifference curves. (The coordinates are x and y). What is the contract curve?

b) (6 marks) Assume now that Bob is risk neutral. Draw Bob's indifference curves in the Edgeworth box. What does risk-neutrality correspond to in standard consumer theory? Find the contract curve and show that Ann is fully insured in every Pareto optimum.

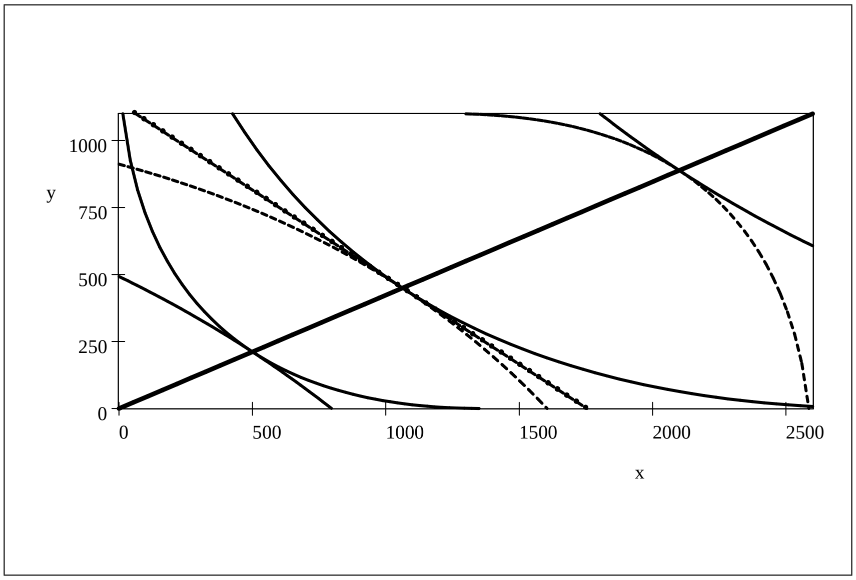
Solution a) Let a denote Ann's expected utility level and b Bob's expected utility level. The indifference curves of the two are then given by the equations

$$\begin{aligned} a &= U(X, Y) = \frac{1}{2}\sqrt{X} + \frac{1}{2}\sqrt{Y} \\ b &= V(X, Y) = \frac{1}{2}\sqrt{2600 - X} + \frac{1}{2}\sqrt{1100 - Y} \end{aligned}$$

or

$$\begin{aligned} y &= \left(2a - \sqrt{x}\right)^2 \\ y &= 1100 - \left(2b - \sqrt{2600 - x}\right)^2 \end{aligned}$$

The following graph shows indifference curves for Ann and Bob in the Edgeworth box.



Along the contract curve both Ann and Bob must have the same slopes of their indifference curves, i.e., they must both have the same marginal rates of substitution. Thus

$$\begin{aligned}
\frac{\partial U}{\partial X} &= \frac{1}{4\sqrt{X}} \qquad \frac{\partial U}{\partial Y} = \frac{1}{4\sqrt{Y}} \qquad MRS_A = \frac{\partial U}{\partial X}\Big/\frac{\partial U}{\partial Y} = \frac{4\sqrt{Y}}{4\sqrt{X}} = \sqrt{\frac{Y}{X}} \\
\frac{\partial V}{\partial X} &= -\frac{1}{4\sqrt{2600-X}} \qquad \frac{\partial V}{\partial Y} = -\frac{1}{4\sqrt{1100-Y}} \\
MRS_B &= \frac{\partial V}{\partial X}\Big/\frac{\partial V}{\partial Y} = \frac{-4\sqrt{1100-Y}}{-4\sqrt{2600-X}} = \sqrt{\frac{1100-Y}{2600-X}}
\end{aligned}$$

The contract curve is characterized by

$$\begin{aligned}
\sqrt{\frac{Y}{X}} &= \sqrt{\frac{1100-Y}{2600-X}} \\
\frac{Y}{X} &= \frac{1100-Y}{2600-X} \\
2600Y - XY &= 1100X - XY
\end{aligned}$$

or

$$Y = \frac{11}{26}X$$

This is the line connecting the allocation $(2600, 1100)$, where Ann gets everything, with the allocation $(0, 0)$, where Bob gets everything. This line is indicated in the graph above. On it typically Ann and Bob share some risk.

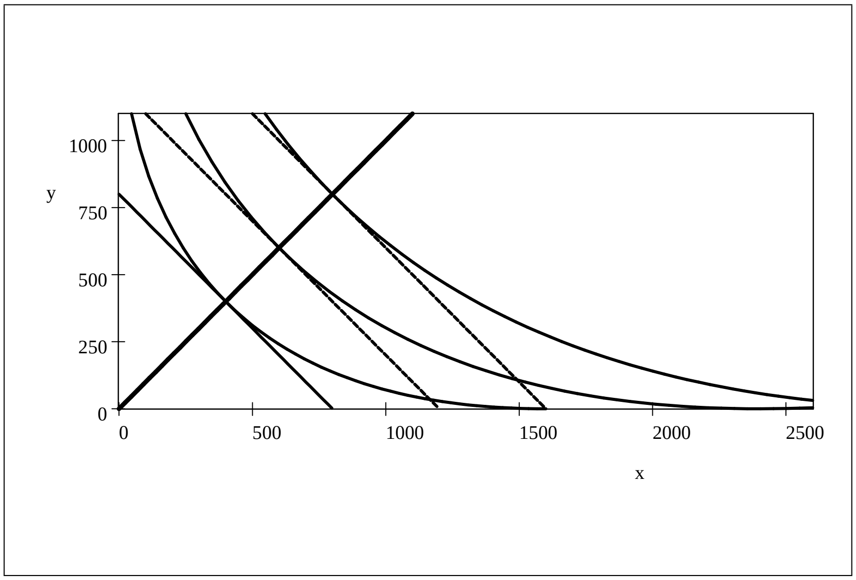
b) Let a denote Ann's expected utility level and b Bob's expected utility level. The indifference curves of the two are now given by the equations

$$\begin{aligned}
a &= U(X,Y) = \frac{1}{2}\sqrt{X} + \frac{1}{2}\sqrt{Y} \\
b &= V(X,Y) = \frac{1}{2}(2600-X) + \frac{1}{2}(1100-Y) = 1850 - \frac{X+Y}{2}
\end{aligned}$$

or

$$
\begin{aligned}
y &= \left(2a - \sqrt{x}\right)^2 \\
y &= 3700 - 2b - X
\end{aligned}
$$

The following graph shows indifference curves for Ann and Bob in the Edgeworth box.



Along the contract curve both Ann and Bob must have the same slopes of their indifference curves, i.e., they must both have the same marginal rates of substitution. Thus

$$
\begin{aligned}
\frac{\partial U}{\partial X} &= \frac{1}{4\sqrt{X}} \qquad \frac{\partial U}{\partial Y} = \frac{1}{4\sqrt{Y}} \qquad MRS_A = \frac{\partial U}{\partial X} \Big/ \frac{\partial U}{\partial Y} = \frac{4\sqrt{Y}}{4\sqrt{X}} = \sqrt{\frac{Y}{X}} \\
\frac{\partial V}{\partial X} &= -\frac{1}{2} \qquad \frac{\partial V}{\partial Y} = -\frac{1}{2} \qquad MRS_B = \frac{\partial V}{\partial X} \Big/ \frac{\partial V}{\partial Y} = 1
\end{aligned}
$$

The contract curve is characterized by

$$
\sqrt{\frac{Y}{X}} == 1
$$

or $Y = X$. This is the "certainty line" where Ann gets the same amount of money regardless of whether an accident occurs or not. Intuitively, Ann doesn't like risk, Bob doesn't mind risk. Unless all the risk is with Bob, both have a reason to trade.

Remark 1 Any Pareto-efficient allocation maximizes Bob's utility subject to the constraint that Ann is getting at least a fixed utility U_0'.

2.2 Alternative formulation

Returning to the insurance-contract model, each contract (P, C) or (x, y) corresponds to a point in a x-y-coordinate system. A contract, if accepted, yields the expected utility

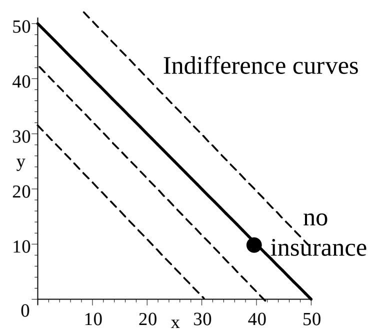
$$
U = (1-p)\, u\left(w^+ - P\right) + pu\left(w^- + C - P\right) = (1-p)\, x + py \tag{2}
$$

to the agent. The agent is indifferent between any two contracts that yield the same expected utility U. His indifference curves over contracts in the x-y-coordinate system are hence given by the lines $(1-p)\,x + py = U$ or

$$y = \frac{U}{p} - \frac{1-p}{p}x \tag{3}$$

for fixed expected utility U. *We see that his indifference curves over contracts are parallel lines with slope* $-\frac{1-p}{p}$. *Higher indifference curves correspond to higher levels of expected utility.* For later use we note that a higher accident probability p yields a lower value of the fraction $\frac{1-p}{p}$ and hence flatter indifference curves.

Using our numeric specifications and $p = p_M = \frac{1}{2}$ his indifference curves are illustrated in the following graph called a *state-space diagram.*



Accepting the insurance contract with premium $P = 0$ and compensation payment $C = 0$ is the same as not getting insured. In this case the agent's utility is $x_0 = u(w^+) = 40$ if no accident occurs and $y_0 = u(w^-) = 10$ if an accident occurs. It will become important later that the point (x_0, y_0) *is below the* $45°$*-degree line* because $w^- < w^+$ and hence $y_0 < x_0$. Notice that the points on the $45°$-degree line $(x = y)$ correspond to the full-insurance contracts. Higher points on this line correspond to lower insurance premia.

If the contract (x, y) is accepted the expected profit to the principal is

$$\Pi = (1-p)\left(w^+ - f(x)\right) + p\left(w^- - f(y)\right) \tag{4}$$

The indifference curve for a given level of expected profits is hence the graph of the function

$$y = u\left(\frac{(1-p)\,w^+ + pw^- - \Pi}{p} - \frac{1-p}{p}f(x)\right)$$

Straightforward calculus shows that the indifference curves are downward sloped and concave.[3] With the above specifications and $p = \frac{1}{2}$ the indifference curves are segments of circles with centre $(0,0)$, as in the figure below.

The following observations will be used throughout. The first implies that lower indifference curves of the principal correspond to higher expected profits.

[3] By the chain rule

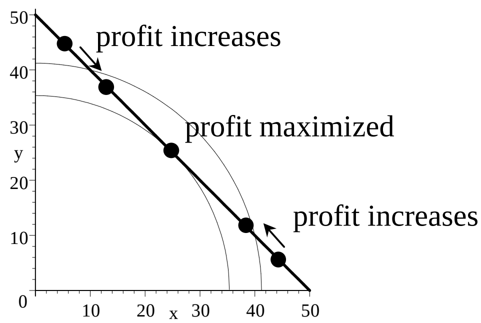
$$\frac{dy}{dx} = u'\left(\frac{(1-p)\,w^+ + pw^- - \Pi}{p} - \frac{1-p}{p}f(x)\right) \times \left(-\frac{1-p}{p}f'(x)\right) < 0$$

Lemma 1 *Reducing the utilities x or y in the the contract (x, y) increases the expected profits of the principal, provided the contract still gets accepted.*

Proof. Reducing x or y increases the expected profit in (4) since f is an increasing function. ■

Fix an indifference curve of the agent. How does the principal's profit vary along this indifference curve of the agent, assuming that the contracts get accepted?

Lemma 2 *Along any fixed indifference curve of the agent the expected profit (4) to the principal is maximized when $x = y$. When $x < y$ an increase in x will increase the expected profits of the principal. When $x > y$ a decrease in x will increase the expected profits.*



Proof. Points (x, y) on the same indifference curve of the agent satisfy Equation (3) with fixed U. The expected profit to the principal is then

$$\pi(x) = (1-p)\left(w^+ - f(x)\right) + p\left(w^- - f\left(\frac{U}{p} - \frac{1-p}{p}x\right)\right)$$

Hence

$$\begin{aligned}
\pi'(x) &= -(1-p) f'(x) - p f'\left(\frac{U}{p} - \frac{1-p}{p}x\right) \times \left(-\frac{1-p}{p}\right) \\
&= -(1-p) f'(x) + (1-p) f'(y) \\
\pi''(x) &= -(1-p) f''(x) - p f''\left(\frac{U}{p} - \frac{1-p}{p}x\right)\left(-\frac{1-p}{p}\right)^2 < 0
\end{aligned}$$

The last equation shows that π is a strictly concave (i.e. downward-bowed) function. If such a function has a critical point it is the unique maximum of the function. Moreover,

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since $u', f' > 0$. The product- and chain rule yield

$$\begin{aligned}
\frac{d^2y}{dx^2} &= u''\left(\frac{(1-p)w^+ + pw^- - \Pi}{p} - \frac{1-p}{p}f(x)\right) \times \left(-\frac{1-p}{p}f'(x)\right)^2 \\
&+ u'\left(\frac{(1-p)w^+ + pw^- - \Pi}{p} - \frac{1-p}{p}f(x)\right) \times \left(-\frac{1-p}{p}f''(x)\right) < 0
\end{aligned}$$

since $u'' < 0$, $u', f', f'' > 0$.

$\pi(x)$ will be strictly increasing to the left of this critical point and decreasing to right. A critical point satisfies $\pi'(x) = 0$ or, equivalently,

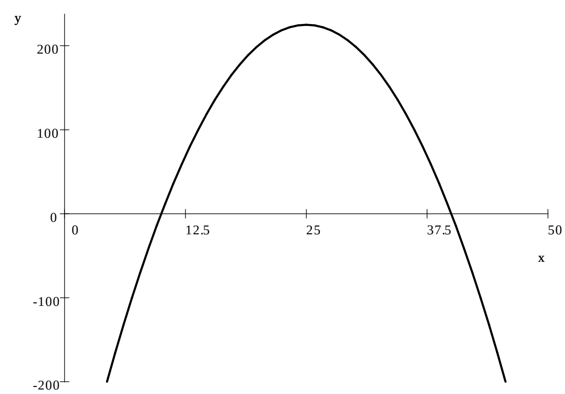
$$(1-p) f'(x) = (1-p) f'(y)$$

with y given by (2). Since f' is an increasing function this implies $x = y = U$, which proves the claim. ■

Example 1 In our numeric example the expected utility of the agent if he does insure himself is $U_0 = (1-p)x_0 + py_0 = 0.5(40+10) = 25$. His indifference curve through (x_0, y_0) is given by $y = \frac{25}{0.5} - \frac{0.5}{0.5}x = 50 - x$. The expected profit to the principal along this indifference curve is hence

$$\begin{aligned}\pi(x) &= 0.5(1600 - f(x)) + 0.5(100 - f(50-x)) \\ &= 0.5\left(1600 - x^2\right) + 0.5\left(100 - (50-x)^2\right) \\ &= -x^2 + 50x - 400\end{aligned}$$

The graph of $\pi(x)$ is



It has a maximum at $x = 25$ where the agent is fully insured since $y = 50 - x = 25$. Notice also that the profit is zero when the agent is uninsured, i.e. when $x = x_0 = 40$ and $y = y_0 = 10$.

Remark 2 The intuition behind the lemma is as follows: Start with a contract (x, y) and $x > y$ on an indifference curve of the agent. If we decrease the agent's utility x if no accident occurs by one unit and increase his utility y if no accident occurs by $\frac{1-p}{p}$ his expected utility will not change and we remain on the agent's indifference curve: The agent loses one unit of utility with probability $(1-p)$ and gains $\frac{1-p}{p}$ units with probability p, overall his expected utility changes by

$$-(1-p) \times 1 + p \times \frac{1-p}{p} = 0$$

$f'(x)$ is the principal's marginal monetary costs of increasing the agent's utility by 1 unit or the marginal gain of reducing his utility by one unit. Hence, by decreasing the agent's utility by one unit if no accident occurs the principal gains in expectation the profit $pf'(x)$. Increasing his utility by $\frac{1-p}{p}$ units if an accident occurs costs the principal in expectation $p \times \frac{1-p}{p} f'(y)$. The principal's expected profit changes overall by

$$(1-p) f'(x) - p \times \frac{1-p}{p} f'(y) = (1-p)(f'(x) - f'(y))$$

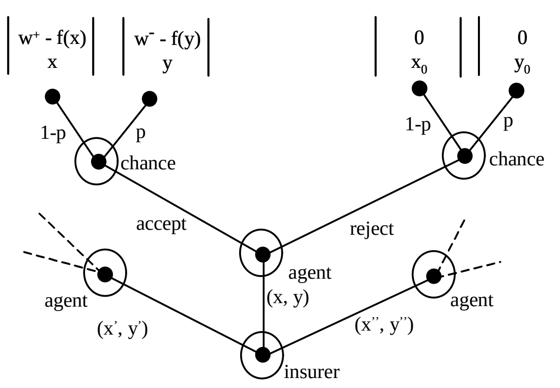
Since f' is increasing, the expected profit increases if $x < y$. There is no change if $x = y$ and if $x < y$ a reduction of x by one unit coupled with an increase of y by $\frac{1-p}{p}$ units would increase the principal's profit.

3 The game

The principal and the agent must now agree on an insurance contract (x, y). The standard procedure assumed in a large body of the contract literature is that the principal makes a take-it-or-leave-it offer and then the agent can accept or reject the contract. This is one of the simplest bargaining models possible. More precisely, the time-structure of this perfect-information game where the principal has infinitely many moves is as follows:

1. The principal offers a contract (x, y) (or (P, C)) to the agent.
2. Observing the offer, the agent can either accept or reject the offer. If he accepts, he has to pay the premium P.
3. A random move decides whether an accident occurs or not.
4. (payoffs are determined)
   (a) If the agent rejects he is uninsured. If no accident occurs the agent receives the net utility x_0 and otherwise y_0. We assume that the principal receives the same net profit 0 in both cases.
   (b) Suppose the agent has accepted the contract. If no accident occurs the agent receives the net utility x. The principal receives the premium and hence his profit is $P = w^+ - f(x)$. If an accident occurs the agent receives net utility y. The principal has to pay the partial compensation C for the damages. His profit is $P - C = w^- - f(y)$, actually a loss.

The principal has to choose among infinitely many contracts (x, y). The following graph shows an excerpt from the game tree.



Remark 3 We did not allow the principal to refuse to offer a contract. However, we allowed him to offer an "insurance contract" with a premium of, e.g. £1.000.000 and a "compensation" of £0. This has the same effect because a rational agent will surely not accept such a contract.

Remark 4 As we will see below the chosen bargaining procedure gives the principal a huge first-mover advantage. He can reap off any gains from trade. This is usually justified with some hand-waving by claiming that the principal is a monopolist while the agent is one of many competing customers for insurance.

Remark 5 Suppose a contract was agreed upon and an accident occurred. The insurer than has to pay the compensation to the agent. Shouldn't we have modelled this explicitly as another move in the game? But if the insurer has the option to pay the money, surely he would also have the option not pay the money. By backward induction, he would then not pay the money since he is then clearly better off. And this is a real danger because, as we all know, it is not easy to get money from insurance companies. If the agent expect this, he would not pay an insurance premium. But this means that no payments are made at all. Consequently backward induction predicts the breakdown of the insurance system! Insurance would not work if the two parties would not have to pay the amounts agreed upon in the insurance contract. Of course, our model does not capture the full story. If the agent does not get his compensation he can write complaints and eventually appeal to a court who may punish the insurer etc. etc. In brief: By not modelling the transactions required by the contract explicitly as moves we avoid the problem why the parties are committed to actually make these transactions. This is an acceptable simplification only when we can assume a well-functioning court system to enforce civil contracts.

4 The optimal insurance contract

...is now determined by backward induction. However, there will be a technical difficulty because the principal has a continuum of moves. Mathematically, we are led to a constrained optimization problem where the principal's profit is to be maximized subject to the participation constraint of the agent.

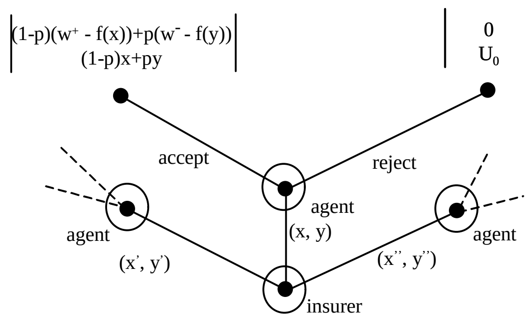
4.1 The random moves

The last decision nodes of the extensive game are the random moves. The backward induction procedure requires us to replace these with terminal nodes where the payoffs are the expected payoffs from the random moves. We have already determined these in the previous calculations. If no insurance is agreed upon the agent gets expected utility $U_0 = (1-p)\,x_0 + py_0$ and the principal gets the expected profit 0. If the contract (x, y) is agreed upon the agent gets the expected utility $(1-p)\,x + py$. The principal gets the expected profit

$$(1-p)\left(w^{+} - f(x)\right) + p\left(w^{-} - f(y)\right) = \left[(1-p)\,w^{+} + pw^{-}\right] - \left[(1-p)\,f(x) + pf(y)\right]$$

Notice that in the employment-contract model $[(1-p)w^+ + pw^-]$ would be the expected revenue the risky task yields. It does not depend on the wages paid. Maximizing expected profits is hence the same as minimizing the expected wage costs $[(1-p)f(x) + pf(y)]$. Also in the insurance-contract model maximizing expected profits is the same as minimizing the expression $[(1-p)f(x) + pf(y)]$.

The reduced game tree is as follows:



4.2 Accept or reject?

Now we have to consider the agent's decision whether to accept the offered insurance contract (x, y) or not. If the agent does not accept the contract he is uninsured and his expected utility is $U_0 = (1-p)x_0 + py_0$. If he accepts the contract, his expected utility is $(1-p)x + py$. He gains from the contract if and only if the *participation constraint*

$$(1-p)x + py \geq U_0$$

of the agent is holds. The name for this inequality should be self-explanatory. We say that the *participation constraint is binding* for a contract (x, y) if it holds with equality, i.e. if

$$(1-p)x + py = U_0$$

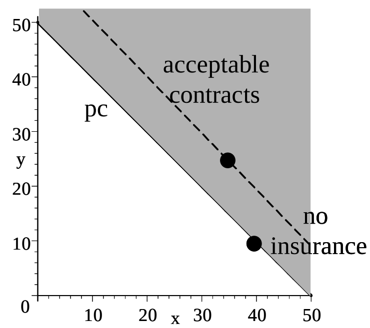
Backward induction yields for a rational agent:

Lemma 3 *a) The agent will accept the contract (x, y) if the participation constraint holds with strict inequality.*

b) The agent will be indifferent between acceptance or rejection of the contract (x, y) if the participation constraint is binding.

c) The agent will reject the contract (x, y) if the participation constraint is violated.

The line marked 'pc' in the following graph shows for which contracts the participation constraint is binding. Contracts above will be accepted, contracts below will be rejected by a rational agent.



4.2.1 A problem with the backward-induction procedure in games with a continuum of moves

For contracts on the line ‘pc’ the agent may accept, reject or randomise between these two decisions. If this situation occurs in a finite game with perfect information we can fix the agent’s choice in anyway we like and then continue with the backward induction procedure to obtain a subgame-perfect equilibrium of the whole game. However, in the current model the principal has a continuum of moves. Hence we cannot fix the agent’s choices after the various contracts that could be offered in any way consistent with rationality and then expect to find, given this behaviour of the agent, an optimal choice for the principal.

To see the difficulty consider the following trivial example: A player has to name a number $0 \leq x \leq 1$. If he names zero, he gets payoff zero. If he names $x > 0$ he gains the payoff $1 - x$. The player’s optimization problem is thus to name the smallest number bigger than zero. If he can only name numbers with a precision of up to three decimals, he has a clear optimal strategy: he should name 0.001. In this case the game is finite. The player has 1001 strategies, $0,000$, 0.001, 0.002, ..., 0.999, 1.000. However, once we allow him to name *any* number there is no optimal strategy since naming $0.000, 1$ is better than naming 0.001, naming $0.000, 001$ is better than naming $0.000, 1$ etc.

The same problem arises in our model when we fix the agent’s behaviour as follows: He accepts any contract if it yields an expected utility strictly above U_0. Otherwise he rejects. Clearly, the behaviour of the agent is rational and consistent with backward induction. However, given this behaviour of the agent there is no optimal strategy for the principal: Suppose there exists some contract which yields a higher expected utility than U_0 to the agent and a strictly positive expected profit to the principal. Take any contract (x, y) with these properties. Since $(1-p)\,x + py > U_0$ we can find a small number $\delta > 0$ such that $(1-p)\,(x-\delta) + p\,(y-\delta) > U_0$, i.e. $\delta = \frac{(1-p)x+py-U_0}{2}$. The contract $(x-\delta, y-\delta)$ would still be accepted by the agent and yield, according to Lemma 1, a higher expected profit than (x, y) to the principal. Hence it cannot be optimal to offer any contract (x, y) that would be accepted and yield positive expected profits. It cannot be optimal either to offer a contract that yields non-positive expected profits and is accepted or a contract that is rejected and hence yields zero expected profits because the principal can guarantee himself a strictly positive profit. Hence an optimal choice for the principal does not exist.

Using continua of strategies is natural in many economic applications and allows us to use the optimization techniques from calculus. However, it can endanger the existence of game theoretic solutions, as in our trivial example. In our insurance model, subgame perfect equilibria still exist, but we cannot follow the backward induction procedure blindly and fix the agent's behaviour without considering whether his behaviour is consistent with an overall solution.

4.3 The optimal contract

The contract proposed in a subgame perfect Nash equilibrium coincides with the contract which maximizes the principal's expected profits subject to the participation constraint of the agent, namely:

Theorem 1 *a) In any subgame perfect equilibrium of the insurance-contract model the principal offers and the agent accepts the contract* (x^*, y^*) *which*
i) fully insures the agent, i.e. $x^* = y^*$*, and*
ii) leaves the agent indifferent between insurance and no-insurance i.e.

$$(1-p)\,x^* + py^* = U_0 = (1-p)\,x_0 + py_0$$

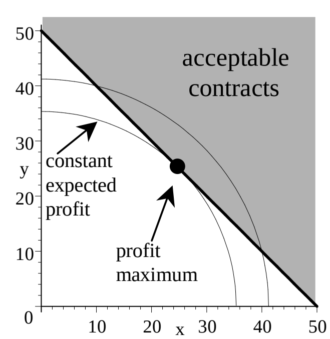
b) A subgame perfect equilibrium with the properties mentioned in a) exists.

Thus, in terms of the primitives of the model, the optimal contract is simply determined by the equation

$$x^* = y^* = (1-p)\,u\left(w^+\right) + pu\left(w^-\right)$$

In our numeric examples $x^* = y^* = 20$ when $p = p_H = \frac{2}{3}$, $x^* = y^* = 25$ when $p = p_M = \frac{1}{2}$ and $x^* = y^* = 30$ when $p = p_L = \frac{1}{3}$. The compensation C is £1500 and the premium $P = £1200$, $P = £975$, or $P = £700$, respectively.

The underlying optimization problem is illustrated in the following graph.



Proof. ad b) (existence) It is a subgame-perfect equilibrium if the principal offers the full insurance contract and the agent accepts any contract that satisfies the participation constraint. As discussed, this behaviour of the agent is optimal. It remains to check the optimality of the principal's offer. Notice first that the "no-insurance" contract (x_0, y_0) would be accepted, but yield expected profit zero. Since the participation constraint is binding for (x_0, y_0), since $y_0 < x_0$ and since (x_0, y_0) yields zero profits, it follows by Lemma

2 that the full insurance $x^* = y^* = U_0$ yields a strictly positive expected profit and yields the highest profit among all contracts for which the participation constraint is binding. (Recall that all these contracts would be accepted) Offering a contract (x, y) where the agent's participation constraint holds with strict inequality cannot be optimal by Lemma 1 because the principal could offer a contract with slightly lower x and y that would still be accepted by the agent and yield a higher profit to the principal. Offering a contract which violates the agent's participation constraint would lead to rejection hence expected profit zero. Overall, the full-insurance contract is the optimal offer.

ad a) (Uniqueness of the subgame perfect equilibrium path) Suppose there is a subgame perfect equilibrium where the full-insurance contract $(x^*, y^*) = (U_0, U_0)$ is not offered with certainty in equilibrium. Let π be the principal's expected profit in this equilibrium.[4] By Lemmas 1 and 2 the full insurance contract, if accepted, yields a higher expected profit than any other contract satisfying the participation constraint, if accepted. In the previous step we have also shown that it yields a strictly positive expected profit. Hence π must be strictly less than the expected profit the full-insurance contract would yield if it were accepted. However, the expected profit in formula (4) depends continuously on x and y. Therefore we can find a small number $\delta > 0$ such that the full-insurance contract $(x, y) = (U_0 + \delta, U_0 + \delta)$, if accepted, yields a strictly higher payoff than π. Since it satisfies the participation constraint with strict inequality, it would be accepted. Therefore, by deviating from the supposed equilibrium and offering $(U_0 + \delta, U_0 + \delta)$ instead the principal could improve his expected payoff. This contradicts the assumption that we started with a subgame perfect equilibrium. ■

Remark 6 In the optimal contract $(x^*, y^*) = (U_0, U_0)$ the expected profit to the principal is

$$\begin{aligned} \Pi &= (1-p)\left(w^+ - f(U_0)\right) + p\left(w^- - f(U_0)\right) \\ &= (1-p)\,w^+ + pw^- - f(U_0) \end{aligned}$$

This is exactly the risk premium of the lottery the agent is facing when uninsured.

The risk premium is the difference between the expected monetary value of the lottery $(1-p)\,w^+ + pw^-$ and its certainty equivalent. The certainty equivalent is the amount of money that, if offered to the agent for sure, would make him indifferent between accepting the money and playing the lottery. If he "plays the lottery" his expected utility is $U_0 = (1-p)\,u(w^+) + pu(w^-)$. The certain amount of money c that yields the same utility must satisfy $u(c) = U_0$ or $c = f(U_0)$. Thus $f(U_0)$ is the certainty equivalent and hence the expected profit Π is the risk premium, as stated.

Remark 7 In our numerical example the principal's expected profit is

$$\begin{aligned} \frac{1}{3}\times 1600 + \frac{2}{3}\times 100 - 20^2 &= 200 \qquad \text{if } p = p_H = \frac{2}{3} \\ \frac{1}{2}\times 1600 + \frac{1}{2}\times 100 - 25^2 &= 225 \qquad \text{if } p = p_M = \frac{1}{2} \\ \frac{2}{3}\times 1600 + \frac{1}{3}\times 100 - 30^2 &= 200 \qquad \text{if } p = p_L = \frac{1}{3} \end{aligned}$$

[4] We do not rule out here that the principal offers a contract which is rejected or that he randomizes between different offers.

5 Exercises

Exercise 1 A risk-averse agent with von Neumann - Morgenstern utility function $u(w) = \sqrt{w}$ faces the risk of an accident which would reduce his wealth from £1600 to £100. The probability of an accident is p. (We are particularly interested in the cases $p_M = \frac{1}{2}$, $p_H = \frac{2}{3}$ and $p_L = \frac{1}{3}$).

i) (1 mark) What is the expected utility of the agent if he does not buy insurance for $p_M = \frac{1}{2}$, $p_H = \frac{2}{3}$ and $p_L = \frac{1}{3}$?

ii) (3 marks) Write down the participation constraint for the three cases.

The set of contracts which yield the same expected profit $\Pi > 0$ to the principal satisfy the equation

$$(1-p)\left(w^+ - x^2\right) + p\left(w^- - y^2\right) = \Pi \qquad \text{or} \qquad y = \sqrt{\frac{c}{p} - \frac{1-p}{p}x^2} \tag{5}$$

where $c = (1-p)\,w^+ + pw^- - \Pi$ and $0 \leq x \leq \sqrt{\frac{c}{1-p}}$.

iv) (3 marks) Draw with some care the graph of the function (5) for $p = p_M = \frac{1}{2}$ and for the values $c = 850$ and $c = 625$. Indicate which curve corresponds to lower expected profits. Based on this graph, which insurance contract do you expect to be optimal for the insurer? What is the expected profit of the principal in this contract.

v) (4 marks) Use the results form the text to determine the optimal contract for the values $p = \frac{2}{3}, \frac{1}{2}$ and $\frac{1}{3}$. What is the expected profit of the principal in the optimal contract?

Exercise 2 Consider next the employment-contract model as sketched in the text.

a) (3 marks) Sketch the extensive game for this game. Let U_0' hereby denote the reservation utility which the agent gets if he does not accept the employment contract. Also assume that the principal makes zero profit if the agent does not accept the wage contract (x, y).

b) (4 marks) The same analysis as for the insurance-contract model shows that it is optimal for the principal to offer, if he wants to hire the agent, the fixed-wage contract (x^{**}, y^{**}) that makes the agent indifferent between acceptance and rejection. For the specifications $u(w) = \sqrt{w}$, $w^+ = 1600$, $w^- = 100$ and $p = \frac{1}{2}$ determine this contract when the agent's reservation utilities are $U_0' = 20$, $U_0' = 25$, $U_0' = 29$ or $U_0' = 35$. In each case determine the fixed wage to be paid to the agent and the principal's expected profit. Will it always be optimal for the principal to hire the agent?

c) (2 marks) Find a general condition under which the principal wants to hire the agent.

Exercise 3 We modify the employment-contract model from the previous exercise as follows, using the particular specification in part b). The principal does not know for sure the reservation utility of the agent. However, he considers only two possibilities. He believes with probability ρ that the reservation utility is U_0'. With the remaining probability $1-\rho$ he believes that it is U_0''.

a) Sketch an extensive game for the model similar to the one in the next handout. Assume that nature determines after the contract has been offered whether the agent has a high or a low reservation utility.

b) Which fixed-wage contract should the principal offer if $U_0' = 25$ and $U_0'' = 35$?

c) Which fixed-wage contract should the principal offer if $U_0' = 20$ and $U_0'' = 25$? (The correct answer depends on ρ.)