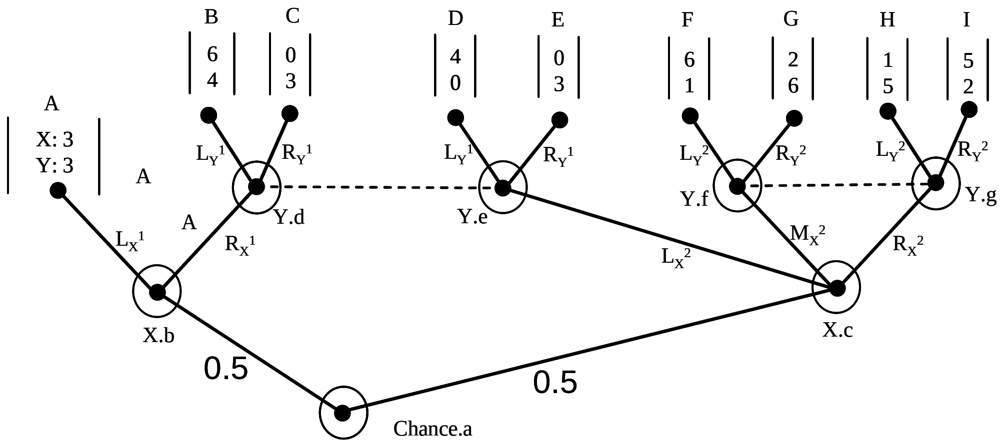
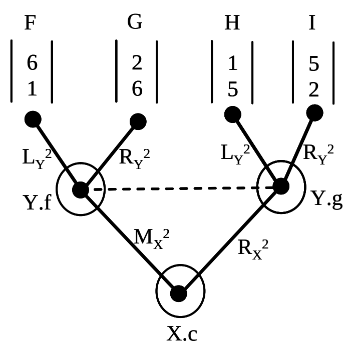
BEE3029 – Economic Issues	Dieter Balkenborg
Solving the signalling model	Departments of Economics
used in the experiment	University of Exeter



We first solve the following quasi-subgame

$X\backslash Y$	L_Y^2	R_Y^2
M_X^2	1 6	6 2
R_X^2	5 1	2 5



This is a game with no Nash equilibria in pure strategies and a unique equilibrium in mixed strategies. We find this equilibrium by determining the mixing probabilities $p = \text{prob}\left(R_X^2\right)$ and $q = \text{prob}\left(R_Y^2\right)$ which make both players indifferent.

Player X will be indifferent between M and R if

$$
\begin{aligned}
& 6\,(1-q) + 2q = (1-q) + 5q \\
\Leftrightarrow\ & 5\,(1-q) = 3q \\
\Leftrightarrow\ & 5 = 8q \\
\Leftrightarrow\ & q = 5/8
\end{aligned}
$$

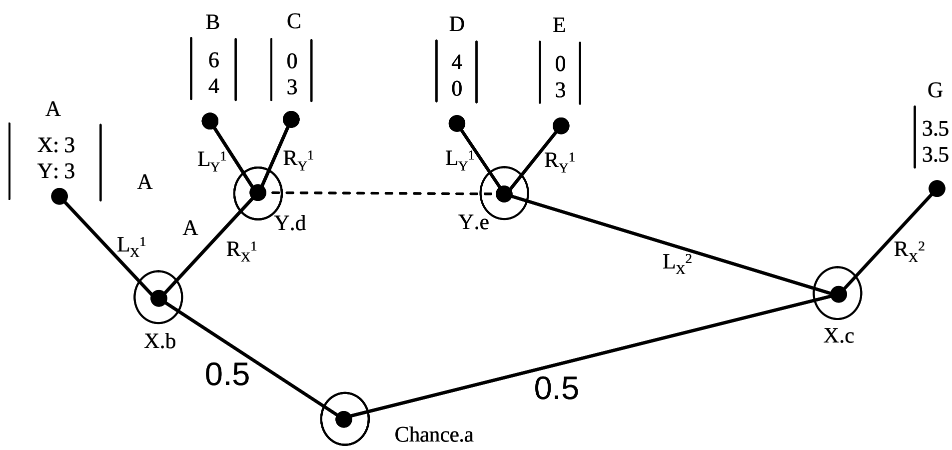
If Y mixes with this probability, player X will get in expectation $6 \times \frac{5}{8} + 2 \times \frac{3}{8} = 1 \times \frac{5}{8} + 5 \times \frac{3}{8} = \frac{20}{8} = 3.5$ regardless of what she does. Similarly, Y will be indifferent between L

and R if

$$\begin{aligned} & (1-p)+5q = 6\,(1-p)+2p \\ \Leftrightarrow\ & 3p = 5\,(1-p) \\ \Leftrightarrow\ & 8p = 5 \\ \Leftrightarrow\ & p = 3/8 \end{aligned}$$

and his expected payoff will also be 3.5.

Thus it suffices to solve the game



Let r denote for this game the probability with which player Y chooses R in the decision situation (information set) $\{d, e\}$. Let p be the probability with which player X chooses R at decision node b and let q be the probability with which she chooses R at decision node c.

Player X will be indifferent at node b if

$$3 = 6\,(1-r) \Leftrightarrow 3r = 6 \Leftrightarrow r = \frac{1}{2}$$

and at c if

$$3.5 = 4\,(1-r) \Leftrightarrow 4r = 0.5 \Leftrightarrow r = \frac{1}{8}$$

Overall, in an optimal response to Y's behaviour, player X should choose

at a	at c	if
R	L	$r < \frac{1}{8}$
R	anything	$r = \frac{1}{8}$
R	R	$\frac{1}{8} < r < \frac{1}{2}$
anything	R	$r = \frac{1}{2}$
L	R	$\frac{1}{2} < r$

We consider only so-called *perfect Bayesian Nash equilibria*, where player Y always maximizes his utility given conditional *beliefs* of being in decision node e (or d) when the information set $\{d, e\}$ is reached. Let μ be the conditional probability with which player

Y believes to be in node e when the information set $\{d, e\}$ is reached. He will be indifferent between L and R at this information set if

$$4(1-\mu) = 3 \Leftrightarrow 4\mu = 1 \Leftrightarrow \mu = \frac{1}{4}$$

He should hence choose at $\{d, e\}$

$$\begin{array}{ccc} L & \text{if} & \mu < \frac{1}{4} \\ \text{anything} & \text{if} & \mu = \frac{1}{4} \\ R & \text{if} & \mu > \frac{1}{4} \end{array}$$

Provided the information set $\{d, e\}$ is reached with positive probability, i.e. if $p+(1-q) > 0$, we can calculate the conditional probability that node e has been reached if $\{d, e\}$ has been reached by Bayes' law as

$$\mu = \frac{1-q}{p+(1-q)}$$

When the information set is not reached, i.e. if $p = 0$ and $q = 1$ then μ cannot be calculated by Bayes' formula and in this case the concept of a perfect Bayesian equilibrium allows to choose μ in any possible way.

Let us for any possible μ determine whether a perfect Bayesian equilibrium with this belief exists. Suppose we have such an equilibrium for $\mu < 1/4$. Given his belief player y must then choose L with certainty ($r = 1$). Therefore player X must play R at node b ($p = 1$) and L at node c ($q = 0$). Thus $\mu = \frac{1}{1+1} = \frac{1}{2} > \frac{1}{4}$, a contradiction.

Suppose $\mu = 1/4$ in the equilibrium. Then Y can randomize in any possible way. However, player Y would never randomize if the opponent does not randomize, except for (L_X^1, R_X^2), when the information set $\{d, e\}$ is not reached. If the information set is reached we must therefore have $r = \frac{1}{8}$ or $r = \frac{1}{2}$. If $r = \frac{1}{2}$ then $q = 1$ and therefore $\mu = 0$, in which case player Y must choose L, a contradiction.

If $r = \frac{1}{8}$ then $p = 1$ while q can take any value. In particular we can choose q such that

$$\mu = \frac{1-q}{1+(1-q)} = \frac{1}{8} \Leftrightarrow 8(1-q) = 1 + (1-q) \Leftrightarrow 7(1-q) = 1 \Leftrightarrow q = \frac{6}{7}$$

Indeed, it is a perfect Bayesian equilibrium if player X always chooses R at node b, chooses R with probability $\frac{7}{8}$ when node c is reached and when player Y chooses R with probability $\frac{1}{8}$ at his information set when it is reached.

For $\mu > \frac{1}{8}$ player Y must choose right with probability 1 and therefore player X must choose left at node b and right at node c. This gives examples of equilibria where the information set $\{c, d\}$ is not reached and so the belief μ cannot be determined by Bayes' law.

The complete set of equilibria with $p = 0$ and $q = 1$ is given by a) $\mu = \frac{1}{4}$ and $r \geq \frac{1}{8}$ and b) $\mu > \frac{1}{8}$, and $r = 1$.

We have found all perfect Bayesian equilibria of the game.