BEE3028 – Economic Issues Lecture	Dieter Balkenborg Departments of Economics University of Exeter
Adverse Selection	

1 Adverse Selection

In this section we return to the insurance model, which differs from the employment model only in respect to what happens when the agent does not accept the contract. In the employment model he gets then an exogeneously fixed reservation utility U_0' from being employed elsewhere (which we should now allow to be "type dependent") whereas he gains in the insurance model the expected utility

$$U_0 = (1-p)\,x_0 + py_0 = (1-p)\,u\left(w^+\right) + pu\left(w^-\right)$$

from being uninsured.

To recall, a monopolistic and risk-neutral insurer (the "principal") seeks to insure in a profit-maximizing way a risk-averse agent against the possibility of an accident. He knows that the agent is risk averse, in fact he knows that the agent's von Neumann-Morgenstern utility function in wealth is $x = u(w)$.[1] We assume that this utility function $u(x)$ is twice differentiable, strictly increasing ($u' > 0$) strictly concave ($u'' < 0$) with a domain of the form $c \leq w < +\infty$ or $c \leq w < +\infty$ where $c \geq -\infty$.[2] In more specific examples we work with the utility function $x = u(w) = \sqrt{w}$. Let $w = f(x)$ denote the inverse function of $u(x)$, so $w = f(x) \Leftrightarrow x = u(w)$. (In particular $f(x) = x^2$ when $u(x) = \sqrt{w}$).

It is also known that the agent's initial wealth is w^+ and that the accident would reduce the agent's wealth from w^+ to w^- where $w^+ > w^-$.[3] In particular, when $u(w) = \sqrt{w}$ they must be both non-negative. In the numerical examples we set $w^+ = £1600$ and $w^- = £100$.

In this handout we study the case where the probability p of an accident is the *private knowledge* of the agent. The principal is, in contrast, uninformed and does not know the probability of an accident. For simplicity we assume that he considers only two possibilities: He thinks that the probability of an accident can be either high ($p = p_H$) or or low ($p = p_L$). To be specific, let $p_H = \frac{2}{3}$ and $p_L = \frac{1}{3}$.

An agent knowing that his accident probability is low will behave differently from an agent who knows that his accident probability is high. The principal may hence be facing two *types* of agents: A LOW-RISK TYPE or a HIGH-RISK TYPE. For expected utility maximization the principal has to have some assessment about the probabilities with which he is facing which type. We assume that he believes to be facing the low-risk type with probability $0 \leq \rho \leq 1$ and the high-risk type with probability $1 - \rho$.

[1] In game theory the model is assumed to be common knowledge.

[2] This includes the cases where the utility function is defined for all numbers on the number line, as for $u(w) = -e^{-w}$, on all non-negative numbers, as for $u(w) = \sqrt{w}$, or on all positive numbers, as for $u(w) - \ln w$.

[3] Of course, both levels of wealth must be in the domain of the utility function since otherwise the expected utility of the agent when he is uninsured would not be defined.

If the principal would know the agent's type, matters would be easy. If he knew that he is facing a low-risk type he would offer him the full insurance contract (x, y) with

$$x = y = u_L^0 = (1 - p_L)\, x_0 + p_L y_0$$

where u_L^0 is the expected utility of the agent if he remain uninsured and, as before, $x_0 = f(w^+) = 40$, $y_0 = f(w^-) = 10$. In our numerical example, the expected utility of the low-risk type if he does not insure himself is $u_L^0 = \frac{2}{3} \times 40 + \frac{1}{3} \times 10 = 30$. The low risk type is hence indifferent between having the wealth $(u_L^0)^2 = 30^2 = £900$ for certain and being uninsured. In the principal's optimal insurance contract the agent pays hence a fee of $P = £700$ and receives the compensation $C = £1500$ if the accident occurs. Still assuming an agent with a low accident probability, the principal's expected profit would be

$$\frac{2}{3} \times £700 + \frac{1}{3}\,(£700 - £1500) = £200$$

The point is, of course, that the agent could be a high-risk type. If the high-risk type accepts the above contract, he ends up for certain with the wealth £900 and hence utility 30 whereas his expected utility is

$$u_H^0 = (1 - p_L)\, x_0 + p_L y_0 = \frac{1}{3} \times 40 + \frac{2}{3} \times 10 = 20$$

if he remained uninsured. So he would certainly accept the above contract if it were offered. However, if the above contract is sold to a high-risk type the principal's expected profit would be

$$\frac{1}{3} \times £700 + \frac{2}{3}\,(£700 - £1500) = -£300$$

Since the principal believes to face the high risk with probability ρ his ex-ante expected profit if he offered the above contract would be

$$(1 - \rho) \times £200 - \rho \times £300 = £\,(200 - 500\rho)$$

Thus, if the principal believes that he is facing the high-risk type of the agent with a probability ρ of more than $2/5 = 40\%$, he would never offer the above contract because he would expect to make a loss. This is the adverse selection problem. The contract which would (from the monopolist's perspective) optimally insure the low-risk type would be bought from the types with the high accident risk.

Of course, if the principal knew the type of the agent, he could price discriminate. Then he would fully insure the high-risk type for a premium of £1200 and the low-risk type for a premium of £700. (Why?)

Because there is asymmetric information the insurance contract cannot condition on the type of the agent. However, the principal can offer a menu of contracts and let the agent choose from it. Having the option to offer a menu of contracts rather than a single contract can only make the principal better off because he can always offer a "menu" consisting only of identical contracts and this is the same as offering a single contract. One can show that it is sufficient to consider only menus with two contracts, since there are only two types. (If there would be three different types of the agent we would have to allow for menus with three contracts etc.) Notice that offering a single contract is a special case where all contracts in the menu are the same. Instead of working with *menus of contract* one can equivalently work with *option contracts*. These are contracts which allow the agent to choose between different options once the contract is signed.

We will denote a menu of two contracts by $((x_L, y_L), (x_H, y_H))$ and consider (x_L, y_L) as the contract designed for the low-risk type to pick and (x_H, y_H) as the contract for the high risk type to pick. At the moment this is labelling only, a more neutral notation like $((x_1, y_1), (x_2, y_2))$ would work as well.

We also use the following bits of notation: If the agent does not insure himself he get the utility $x_0 = f(w)^+$ if no accident occurs and $y_0 = f(w^-)$ if an accident occurs. The expected utility of the uninsured agent is therefore $u_L = (1 - p_L) x_0 + p_L y_0$ if he is of the low-risk type and $u_H = (1 - p_H) x_0 + p_H y_0$ if he is of the high-risk type. (In previous handouts I used the notation U_0.) Clearly, $u_L > u_H$.

We model the monopoly power of the principal by assuming that he can make a take-it-or-leave-it offer to the agent. To obtain a perfect-information game we assume that the agent's type is determined after the principal has offered a menu of contracts. This allows us to solve the model in essence by backward-induction. (Alternatively, one can work with Bayesian perfect equilibrium.) Imagine the principal offers his menu of insurance contracts on the internet. Then next customer visiting his WEB site and buying insurance is with probability ρ of the high-risk type and with probability $1 - \rho$ of the low-risk type.[4] We can now fix the time-structure and the extensive form game as follows:

The extensive form game is indicated in Figure 1.

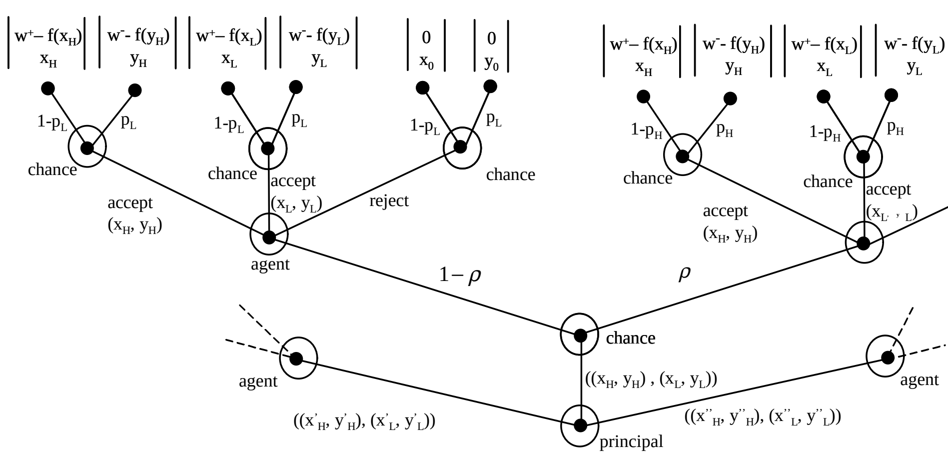

Figure 1

1. The principle offers a menu of contracts $((x_L, y_L), (x_H, y_H))$.

2. Nature determines the type of the agent. With probability ρ_H he is chosen to be a high-risk type and with probability ρ_L he is chosen to be a low-risk type.

3. The agent either chooses an insurance contract form the menu offered or he chooses not to insure himself. (If he accepts one he has to pay the appropriate fee.)

[4] In this interpretation you may think of the different types of the agent as different players.

4. An accident occurs with the relevant probability p_L or p_H.
5. If required by contract, the principal has to pay the appropriate compensation. Final payoffs are determined.

Following the backward induction procedure we obtain Figure 2 by eliminating all decision nodes of nature at the end of the game and replacing them by terminal nodes with the appropriate expected payoffs.

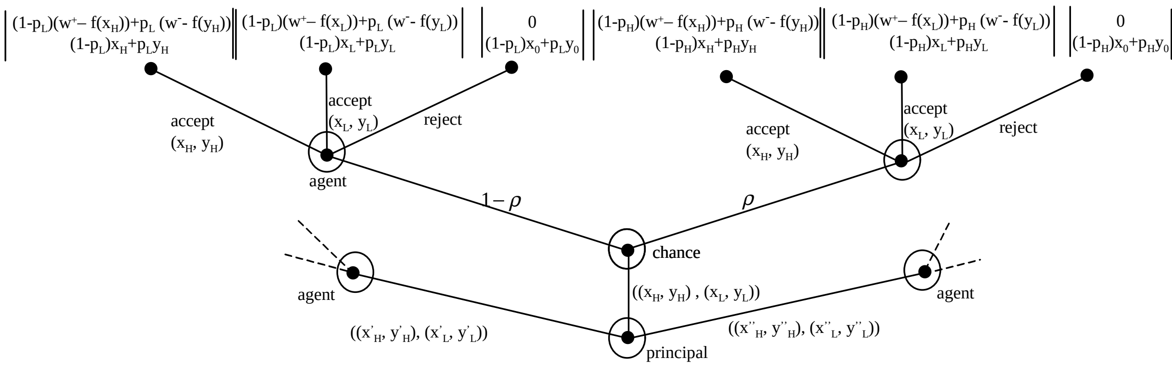

Figure 2

2 The participation- and incentive constraints

Fix a a menu of contracts $((x_L, y_L), (x_H, y_H))$ and consider the subgame after this menu has been offered. If the inequalities

$$
\begin{aligned}
(1-p_L)\,x_L + p_L y_L &\geq (1-p_L)\,x_0 + p_L y_0 = u_L && \text{(PCL)}\\
(1-p_L)\,x_L + p_L y_L &\geq (1-p_L)\,x_H + p_L y_H && \text{(ICL)}\\
(1-p_H)\,x_H + p_H y_H &\geq (1-p_H)\,x_0 + p_H y_0 = u_H && \text{(PCH)}\\
(1-p_H)\,x_H + p_H y_H &\geq (1-p_H)\,x_L + p_H y_L && \text{(ICH)}
\end{aligned}
$$

are satisfied for then it will be a optimal for each type of agent to accept the contract (x_L, y_L) or, respectively, (x_H, y_H) designed for him. However, unless the four constraints are satisfied with strict inequality ($<$) other behavior can also be optimal. For instance, if (PCL) holds with equality the low-risk type of the agent can randomize between accepting (x_L, y_L) and remaining uninsured. Needless to say, if one of the constraints is violated it cannot be optimal for both types of the agent to accept the contract designed for them.

Let us consider these four constraints in more detail. In our graphs, the contract (x_L, y_L) will be indicated by a square, the contract (x_H, y_H) by a circle.

Inequality (PCL) is the *participation constraint* for the low-risk agent, expressing that he is at least as well off by accepting (x_L, y_L) than by remaining uninsured. The inequality (ICL) is the *incentive constraint* for the low-risk type expressing that he is at least as well off by accepting the contract (x_L, y_L) designed for him rather than the contract (x_H, y_H) designed for the high-risk type.

The conditions can be illustrated in a coordinate system (x, y) as follows: The high risk type is indifferent between two contracts (x, y) and (x', y') if

$$(1-p_H)\,x' + p_H y' = (1-p_H)\,x + p_H y \Leftrightarrow \frac{y'-y}{x'-x} = -\frac{1-p_H}{p_H}$$

i.e. both contracts lie on a line with slope $-\frac{1-p_H}{p_H}$. Therefore the indifference curve of the high-risk type through the point (x, y) is the line with slope $-\frac{1-p_H}{p_H}$ through this point. All contracts on this line are as good as (x, y) for the high-risk type, contracts above are strictly preferred and contracts below are worse than (x, y). Consider, for instance, the indifference curve with slope $-\frac{1-p_H}{p_H}$ through the "initial endowment point" (x_0, y_0) of the agent. The participation constraint (PCH) states that the contract (x_L, y_L) designed for the high-risk type must be on or above this line to be acceptable by the agent. In our specific example, both contracts indicated by a circle "◯" and square "□" would be accepted by the high-risk type because they are above the flatter line 'pch'.

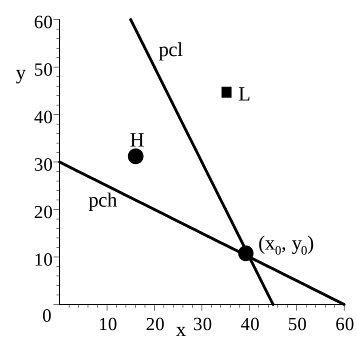

Figure 3

Similarly, since

$$(1-p_L)\,x' + p_L y' = (1-p_L)\,x + p_L y \Leftrightarrow \frac{y'-y}{x'-x} = -\frac{1-p_L}{p_L}$$

the indifference curves of the low-risk type are the lines with slope $-\frac{1-p_L}{p_L}$. Because

$$\begin{aligned} p_L &< p_H \Leftrightarrow p_L - p_L p_H < p_H - p_H p_L \Leftrightarrow p_L\,(1-p_H) < p_H\,(1-p_L) \\ &\Leftrightarrow \frac{(1-p_H)}{p_H} < \frac{(1-p_L)}{p_L} \Leftrightarrow -\frac{(1-p_L)}{p_L} < -\frac{(1-p_H)}{p_H} \end{aligned}$$

the indifference curves of the low-risk type fall more steeply than that of the high-risk type. (Intuitively, the high-risk type values utility in the accident state realtively more than the low-risk type). The line 'pcl' where participation constraint for the low-risk type is binding also goes through the point (x_0, y_0), but is steeper. In the graphic, the low-risk type would accept the contract indicated by a square above the steeper line, but not the one indicated by a circle.

The two incentive constraints inform us about the relative position of the two contracts (x_L, y_L) and (x_H, y_H). Draw the indifference curves for both types through (x_L, y_L). Then

(ICH) requires that (x_H, y_H) is on or above the indifference curve for the high risk-type while (ICL) requires that (x_H, y_H) is on or below the indifference curve for the low-risk type. (x_H, y_H) must hence be in the shaded region of the following Figure 4. Alternatively, draw the indifference curves for the two types through (x_H, y_H). Then (ICL) requires that (x_L, y_L) is on or above the indifference curve for the low-risk type through (x_H, y_H) while (ICH) requires that (x_L, y_L) is on or below the indifference curve for the high risk type through (x_H, y_H). (x_L, y_L) must hence be in the shaded region of the following Figure 5.

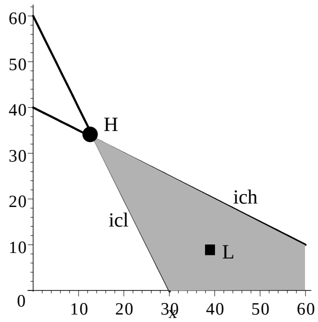

Figure 4

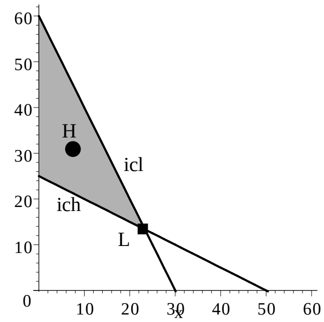

Figure 5

3 Basic structure of optimal menues

Suppose the principal offers the menu $((x_L, y_L), (x_H, y_H))$, the low-risk type of the agent accepts the contract (x_L, y_L) and the high-risk type (x_H, y_H). Recall that $f(x)$ is the amount of money the agent must have such that his net utility is x. The expected profit to the principal is therefore

$$\begin{aligned} & \Pi((x_L, y_L), (x_H, y_H)) \\ = & (1-\rho)\left((1-p_L)\left(w^+ - f(x_L)\right) + p_L\left(w^- - f(x_L)\right)\right) \\ & +\rho\left((1-p_H)\left(w^+ - f(x_H)\right) + p_H\left(w^- - f(x_H)\right)\right) \\ = & (1-\rho)\left((1-p_L)w^+ + p_L w^-\right) + \rho\left((1-p_L)w^+ + p_L w^-\right) \\ & -C((x_L, y_L), (x_H, y_H)) \end{aligned}$$

where the term $(1-\rho)((1-p_L)w^+ + p_L w^-) + \rho((1-p_L)w^+ + p_L w^-)$ is a constant independent of the choice variables x_L, y_L, x_H, y_H of the principal and where

$$\begin{aligned} & C((x_L, y_L), (x_H, y_H)) \\ = & (1-\rho)((1-p_L)f(x_L) + p_L f(y_L)) + \rho((1-p_H)f(x_H) + p_H f(y_H)) \end{aligned}$$

is the expected "cost" to be payed to the agent.

The principal seeks to offer a menu that will maximize his profits. He always has the option to offer the contract (x_0, y_0) in his menu, which is basically the same as "offering" no insurance. Instead of rejecting, the agent could always accept this contract. It hence intuitive that when seeking an optimal menu for the principal we can restrict ourselves to menus such that both types of agents would accepts the contracts designed for them.

Hence we are let to the following *constrained optimization problem:*

(*) Minimize the function $C\left(\left(x_L, y_L\right), \left(x_H, y_H\right)\right)$ subject to the four linear constraints (PCL), (PCH), (ICL), (ICH).

Lemma 1 *The function $C\left(\left(x_L, y_L\right), \left(x_H, y_H\right)\right)$ is strictly convex.*

The matrix of second partial derivatives of C is

$$\begin{bmatrix} (1-\rho)(1-p_L) f''(x_L) & 0 & 0 & 0 \\ 0 & (1-\rho) p_L f''(y_L) & 0 & 0 \\ 0 & 0 & \rho(1-p_H) f''(x_H) & 0 \\ 0 & 0 & 0 & \rho p_H f''(y_L) \end{bmatrix}$$

which is immediately verified to be negative definite since $f'' < 0$.

Let the menu $\left(\left(x_L^*, y_L^*\right), \left(x_H^*, y_H^*\right)\right)$ be a solution to the constrained minimization problem. We will show later that it exists and is unique.

Theorem 1 *a) The game has a subgame perfect equilibrium where the principal offers the menu $\left(\left(x_L^*, y_L^*\right), \left(x_H^*, y_H^*\right)\right)$ solving the constrained optimization problem (*), the low risk type accepts $\left(x_L^*, y_L^*\right)$ and the high-risk type accepts $\left(x_H^*, y_H^*\right)$.*

b) In all other subgame perfect equilibria of the game the principal and both types of the agent get the same payoff as in the equilibrium described in a).

Theorem 2 *In a solution $\left(\left(x_L^*, y_L^*\right), \left(x_H^*, y_H^*\right)\right)$ to the constrained optimization problem (*)*

1. *The high-risk type is fully insured, i.e.*

$$x_H^* = y_H^*.$$

Moreover, $u_H \leq x_H^* \leq u_L$.

2. *The incentive constraint of the high-risk type is binding, i.e.*

$$(1-p_H) x_H^* + p_H y_H^* = (1-p_H) x_L^* + p_H y_L^*$$

3. *The participation constraint of the low-risk type is binding*

$$(1-p_L) x_L^* + p_L y_L^* = (1-p_L) x_0 + p_L y_0 = u_L$$

Proof: The proof is done in four steps, each time by contradiction. We make heavy use of the two principles established in Lemma 1 of the handout on insurance: Provided it does not interfere with the participation and incentive constraints, it increases profits to reduce the utilities in any of the contracts. Moreover, it increases profits to move the contract (x_L, y_L) (respectively (x_H, y_H)) along an indifference curve of the low-risk (respectively high-risk) type.

Step 1: One of the two participation constraints must be binding (i.e. hold with equality) in the optimal menu $((x_L^*, y_L^*), (x_H^*, y_H^*))$.

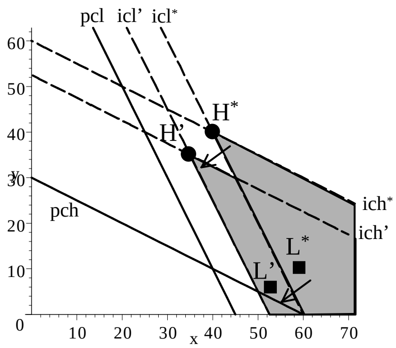

Figure 6

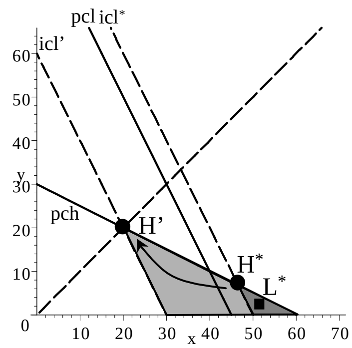

Figure 7

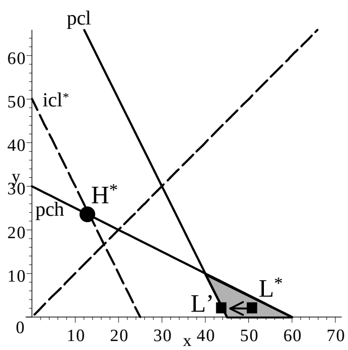

Figure 8

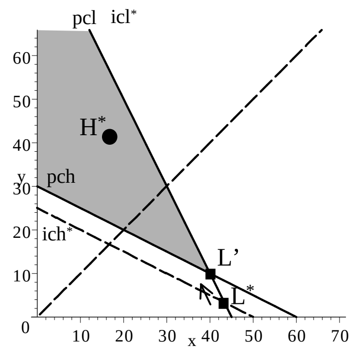

Figure 9

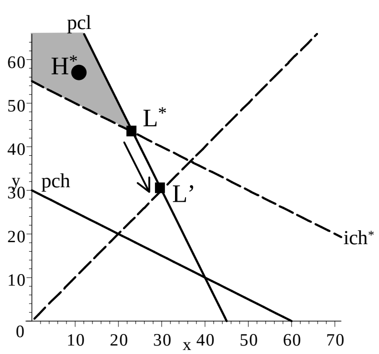

Figure 10

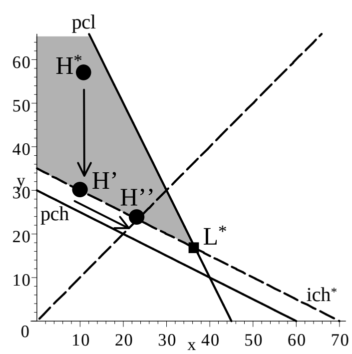

Figure 11

Proof: (See Figure 6) Suppose all four constraints hold with a strict inequality in the optimal menu $((x_L^*, y_L^*), (x_H^*, y_H^*))$. Since all terms in the constraints depend continuously on the variables we can find a small number $\delta > 0$ such that the menu

$$((x_L, y_L), (x_H, y_H)) = ((x_L^* - \delta, y_L^* - \delta), (x_H^* - \delta, y_H^* - \delta))$$

also satisfies all four constraints with strict inequality. However,

$$C((x_L, y_L), (x_H, y_H)) < C((x_L^*, y_L^*), (x_H^*, y_H^*))$$

as one can see when comparing term-by-term because f is strictly increasing. For instance, $f(x_L) = f(x_L^* - \delta) < f(x_L)$ etc. This contradicts the assumption that $((x_L^*, y_L^*), (x_H^*, y_H^*))$ minimizes C subject to the constraints.

Step 2: The participation constraint must be binding for the low-risk type**.**

Proof: Suppose the participation constraint is binding for the high risk type (i.e. (x_H^*, y_H^*) is on the line (pch) in Figures 7), but not for the low risk type (i.e. (x_L^*, y_L^*) is above the line (pcl)). There are two possibilities. a) $x_H^* > y_H^*$, as in Figure 7. Because the incentive constraints must hold, (x_L^*, y_L^*) must be in dark shaded area of the figure. Consider the alternative contract $((x_L^*, y_L^*), (u_H, u_H))$ where we offer full insurance to the high-risk type. All incentive constraints and participation constraints are satisfied in this new contract. The principal gets the same expected profit from the low-risk agent as before. However, his expected profit from the high-risk type is strictly higher (movements along an indifference curve towards full insurance increase the profits).[5] Thus the menu $((x_L^*, y_L^*), (x_H^*, y_H^*))$ cannot have been optimal. b) $x_H^* \leq y_H^*$, as in Figure 8. The constraints (PCL) and (ICH) imply that (x_L^*, y_L^*) is in the shaded region of the figure, but not on the line (pcl). Consider the alternative menu $((x_L, y_L^*), (x_H^*, y_H^*))$ where x_L is obtained by reducing the first component in (x_L^*, y_L^*) until (PCL) becomes binding (indicated as the contract L' in the graph). Again, the new contract satisfies all four constraints. However, the low-risk agent is now paid less when no accident occurs, so $C((x_L, y_L), (x_H, y_H))$ is reduced, again in contradiction to the optimality of $((x_L^*, y_L^*), (x_H^*, y_H^*))$.

Step 3: $x_0 \leq x_L^* \leq u_L$, i.e., (x_L^*, y_L^*) is on the line segment between non-insurance (x_0, y_0) and full insurance (u_L, u_L) for the low risk agent.

Proof: First assume $x_L^* > x_0$ (see Figure 9). This means that if no accident occurs the low-risk type has a higher utility when he is insured than when he is uninsured. The contract requires a negative premium for the low-risk type. (x_H^*, y_H^*) must then be in the shaded region of Figure 9, by (PCH), and (ICL). (ICH is then automatically satisfied.) Then the contract $((x_H^*, y_H^*), (x_0, y_0))$ where the low-risk type is fully insured, still satisfies all four constraints, but yields higher profit to the principal. Contradiction. Suppose next $x_L < u_L$ and therefore $y_L > u_L$. (x_H^*, y_H^*) must be in the shaded region of the Figure 10. This time the contract $((x_H^*, y_H^*), (u_L, u_L))$ also satisfies all constraints and brings higher profit to the principal.

Step 4: (ICH) is binding and the low-risk type is fully insured.

Proof: (x_H^*, y_H^*) must be in the shaded region of Figure 11. First reduce x_H from x_H^* until (ICH) becomes binding. This increases the principal's expected payoff and all constraints remain binding. Then move the contract (x_H, y_H) along the line (ich) until it hits the 45^o-line. Again, the principal's contract increases and all constraints are satisfied. Thus, unless we already started in a contract where (ICH) is binding and the high-risk type is fully insured, the initial menu cannot have been binding.

Steps 3 and 4 imply $u_H \leq x_H^* \leq u_L$.

this concludes the proof of the theorem.

[5]To make sure that both types accept their contracts in the new menu, increase utility in each contract by a small amount, but not so much that the principal loses compared to the previous offer. I will not always point out this type of subtlety in the following.

4 The final step

The previous result gave us the restriction

$$u_H \leq x_H^* \leq u_L \tag{1}$$

to be satisfied by one of the unknowns and the three equations

$$\begin{aligned} y_H^* &= x_H^* \\ x_H^* &= (1-p_H)\, x_L^* + p_H y_L^* \\ u_L &= (1-p_L)\, x_L^* + p_L y_L^* \end{aligned}$$

With the three equations we can eliminate all but one variable, for instance

$$\begin{aligned} y_H^* &= x_H^* \\ x_L^* &= \frac{p_L x_H^* - p_H u_L}{p_L - p_H} \\ y_L^* &= \frac{(1-p_L)\, u_L - (1-p_L)\, x_H^*}{p_L - p_H} \end{aligned} \tag{2}$$

We have not completely solved the problem, but we have reduced a constrained optimization problem with four unknowns to an "almost" unconstrained optimization problem in one variable.[6] Namely, we must still minimize the "cost function"

$$\begin{aligned} & c\left(x_H^*\right) = C\left(\left(x_L^*, y_L^*\right), \left(x_H^*, y_H^*\right)\right) \\ = \; & (1-\rho)\left((1-p_L)\, f\left(x_L^*\right) + p_L f\left(y_L^*\right)\right) + \rho\left((1-p_H)\, f\left(x_H^*\right) + p_H f\left(y_H^*\right)\right) \\ = \; & (1-\rho)\left((1-p_L)\, f\left(\frac{p_L x_H^* - p_H u_L}{p_L - p_H}\right) + p_L f\left(\frac{(1-p_L)\, u_L - (1-p_L)\, x_H^*}{p_L - p_H}\right)\right) \\ & + \rho f\left(x_H^*\right) \end{aligned}$$

$c\left(x_H^*\right)$ is convex since it is obtained from a convex function by linear substitutions of the variables.

The results in the previous section did not depend on the probability ρ with which the principal faces a high-risk type. When calculating the minimum of $c\left(x_H^*\right)$ this is crucial.

In particular, when $\rho = 1$ we have to minimize the function $f\left(x_H^*\right)$ subject to the constraint $u_H \leq x_H^* \leq u_L$. Since $f(x)$ is increasing this minimum occurs when $x_H^* = u_H$. The equations (2) imply then $y_H^* = x_H^*$ and $x_L^* = x_0$, $y_L^* = y_0$. This is as it should be, when there is no low-risk type, the optimal insurance is to fully insure the high-risk type and to offer no insurance to the low-risk type.

Notice that this solution is not an "interior" optimum determined by the first-order condition $c'\left(x_H^*\right) = 0$. The first-order condition has for $\rho = 1$ no solution in the positive numbers.[7] The solution is determined by the constraint $u_H \leq x_H^*$. Using continuity considerations one can show that this constraint must also be binding for all ρ sufficiently close to 1. This means that if the probability of a high-risk type is sufficiently close to

[6] Except that restriction (1) must still hold.

[7] At least when some additional technical assumptions are satisfied.

one, then it is optimal to fully insure the high-risks and offer no insurance to the low-risks. The high risks drive the low risks out of the market.

For $\rho = 0$ the optimization problem boils down to finding the optimal insurance contract for the low-risk type subject to the participation constraint of the low-risk type holding with equality. As we know, the solution is to fully insure the low-risk type. This could mean that a loss is made on the high-risk type. However, those occur with probability zero.

Finally, a more detailed analysis of the model shows that x_H^* is nonincreasing in ρ.

Rather than giving the general analysis it must suffice here to do the calculation in our example. Our restriction is $20 \leq x_H^* \leq 30$. We must have

$$\begin{aligned} y_H^* &= x_H^* \\ x_H^* &= \frac{1}{3}x_L^* + \frac{2}{3}y_L^* \\ 30 &= \frac{2}{3}x_L^* + \frac{1}{3}y_L^* \end{aligned}$$

or

$$\begin{aligned} y_H^* &= x_H^* \\ x_L^* &= 60 - x_H^* \\ y_L^* &= 2x_H^* - 30 \end{aligned} \tag{3}$$

The function to be minimized is

$$\begin{aligned} c\left(x_H^*\right) &= (1-\rho)\left(\frac{2}{3}\left(x_L^*\right)^2 + \frac{1}{3}\left(y_L^*\right)^2\right) + \rho\left(x_H^*\right)^2 \\ &= (1-\rho)\left(\frac{2}{3}\left(60 - x_H^*\right)^2 + \frac{1}{3}\left(2x_H^* - 30\right)^2\right) + \rho\left(x_H^*\right)^2 \end{aligned}$$

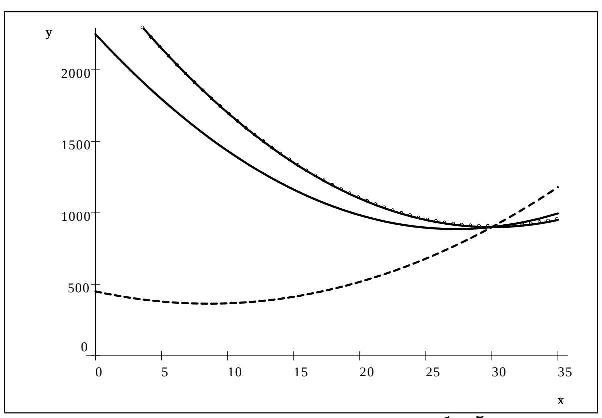

Figure 12: $c\left(x_H^*\right)$ for $\rho = \frac{1}{6}, \frac{5}{6}, 1$

This is a quadratic function in x_H^* with a unique minimum where

$$\begin{aligned} \frac{\partial c}{\partial x_H^*} &= (1-\rho)\left[\frac{2}{3}\left(60 - x_H^*\right)(-1) + \frac{1}{3}\left(2x_H^* - 30\right)(2)\right] + 2\rho x_H^* \\ &= (1-\rho)\left[2x_H^* - 60\right] + 2\rho x_H^* = 2x_H^* - 60(1-\rho) = 0 \\ &\Leftrightarrow\ x_H^* = 30(1-\rho) \end{aligned}$$

The restriction $20 \leq x_H^*$ implies that this solution is only valid when $30\,(1-\rho) \geq 20$ or $\rho \leq \frac{1}{3}$. Thus the final solution is given by

$$x_H^* = \begin{cases} 30\,(1-\rho) & \text{for} \quad \rho \leq \frac{1}{3} \\ 20 & \text{for} \quad \rho > \frac{1}{3} \end{cases}$$

and the equations 3.